The Kelmans-Seymour conjecture II: 2-vertices in K_4^-

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Abstract

We use K_4^- to denote the graph obtained from K_4 by removing an edge, and use TK_5 to denote a subdivision of K_5. Let G be a 5-connected nonplanar graph and $\{x_1, x_2, y_1, y_2\} \subseteq V(G)$ such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ with $y_1y_2 \notin E(G)$. Let $w_1, w_2, w_3 \in N(y_2) - \{x_1, x_2\}$ be distinct. We show that G contains a TK_5 in which y_2 is not a branch vertex, or $G - y_2$ contains K_4^-, or G has a special 5-separation, or $G - \{y_2v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}$ contains TK_5.

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1 Introduction

We use notation and terminology from [3]. In particular, for a graph K, we use TK to denote a *subdivision* of K. The vertices in a TK corresponding to the vertices of K are its *branch vertices*. Kelmans [6] and, independently, Seymour [11] conjectured that every 5-connected nonplanar graph contains TK_5. In [7, 8], this conjecture is shown to be true for graphs containing K_4^-.

In [3] we outline a strategy to prove the Kelmans-Seymour conjecture for graphs containing no K_4^-. Let G be a 5-connected nonplanar graph containing no K_4^-. Then by a result of Kawarabayashi [4], G contains an edge e such that G/e is 5-connected. If G/e is planar, we can apply a discharging argument. So assume G/e is not planar. Let M be a maximal connected subgraph of G such that G/M is 5-connected and nonplanar. Let z denote the vertex representing the contraction of M, and let $H = G/M$. Then one of the following holds:

(a) H contains a K_4^- in which z is of degree 2.

(b) H contains a K_4^- in which z is of degree 3.

(c) H does not contain K_4^-, and there exists $T \subseteq H$ such that $z \in V(T)$, $T \cong K_2$ or $T \cong K_3$, and H/T is 5-connected and planar.

(d) H does not contain K_4^-, and for any $T \subseteq H$ with $z \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, H/T is not 5-connected.

In this paper, we deal with (a) by taking advantage of the K_4^- containing z. We prove the following result, in which the vertex y_2 plays the role of z above.

Theorem 1.1 *Let G be a 5-connected nonplanar graph and $\{x_1, x_2, y_1, y_2\} \subseteq V(G)$ such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ with $y_1y_2 \notin E(G)$. Then one of the following holds:*

(i) G contains a TK_5 in which y_2 is not a branch vertex.

(ii) $G - y_2$ contains K_4^-.

(iii) G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{y_2, a_1, a_2, a_3, a_4\}$, and G_2 is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ and the 4-cycle $b_1b_2b_3b_4b_1$ by adding y_2 and the edges y_2b_i for $i \in [4]$.

(iv) For $w_1, w_2, w_3 \in N(y_2) - \{x_1, x_2\}$, $G - \{y_2v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}$ contains TK_5.

Note that when Theorem 1.1 is applied later, G will be a graph obtained from a 5-connected nonplanar graph by contracting a connected subgraph, and y_2 represents that contraction. So we need a TK_5 in G to satisfy (i) or (iv) to produce a TK_5 in the original graph. Note that (ii) will not occur if the original graph is K_4^--free. Moreover, if (iii) occurs then we may apply Proposition 1.3 in [3] to produce a TK_5 in the original graph.

The arguments used in this paper to prove Theorem 1.1 is similar to those used in [7, 8]. Namely, we will find a substructure in the graph and use it to find the desired TK_5. However, since the TK_5 we are looking for must use certain special edges at y_2, the arguments here are more complicated and make heavy use of the option (ii).

We organize this paper as follows. In Section 2, we collect a few known results that will be used in the proof of Theorem 1.1. We will produce an intermediate structure in G which consists of eight special paths X, Y, Z, A, B, C, P, Q, see Figure 1 (where X is the path in bold and Y, Z are not shown). In Section 3, we find the path X in G between x_1 and x_2 whose deletion results in a graph satisfying certain connectivity requirement. In Section 4, we find the paths Y, Z, A, B, C, P, Q in G. In Section 5, we use this structure to find the desired TK_5 for Theorem 1.1.

2 Previous results

Let G be a graph and $A \subseteq V(G)$, and let k be a positive integer. Let $[k] = \{1, 2, ..., k\}$. Let C be a cycle in G with a fixed orientation (so that we can speak of clockwise and anticlockwise directions). For two vertices $x, y \in V(C)$, xCy denotes the subpath of C from x to y in clockwise order. (If $x = y$ then xCy denotes the path consisting of the single vertex x.) Recall from [3] that G is (k, A)-*connected* if, for any cut T of G with $|T| < k$, every component of $G - T$ contains a vertex from A. We say that (G, A) is *plane* if G is drawn in the plane with no crossing edges such that the vertices in A are incident with the unbounded face of G. Moreover, for vertices $a_1, \ldots, a_k \in V(G)$, we say $(G, a_1, \ldots, a_k)$ is *plane* if G is drawn in a closed disc in the plane with no crossing edges such that $a_1, \ldots, a_k$ occur on the boundary of the disc in this cyclic order. We say that (G, A) is *planar* if G has a plane representation such that (G, A) is plane. Similarly, $(G, a_1, \ldots, a_k)$ is *planar* if G has a plane representation such that $(G, a_1, \ldots, a_k)$ is plane.

In this section, we list a few known results that we need. We begin with a technical notion. A *3-planar graph* $(G, \mathcal{A})$ consists of a graph G and a collection $\mathcal{A} = \{A_1, \ldots, A_k\}$ of pairwise disjoint subsets of $V(G)$ (possibly $\mathcal{A} = \emptyset$) such that

- for distinct $i, j \in [k]$, $N(A_i) \cap A_j = \emptyset$,
- for $i \in [k]$, $|N(A_i)| \leq 3$, and
- if $p(G, \mathcal{A})$ denotes the graph obtained from G by (for each $i \in [k]$) deleting A_i and adding new edges joining every pair of distinct vertices in $N(A_i)$, then $p(G, \mathcal{A})$ can be drawn in a closed disc with no crossing edges.

If, in addition, $b_1, \ldots, b_n$ are vertices in G such that $b_i \notin A_j$ for all $i \in [n]$ and $j \in [k]$, $p(G, A)$ can be drawn in a closed disc in the plane with no crossing edges, and $b_1, \ldots, b_n$ occur on the boundary of the disc in this cyclic order, then we say that $(G, \mathcal{A}, b_1, \ldots, b_n)$ is *3-planar*. If there is no need to specify $\mathcal{A}$, we will simply say that $(G, b_1, \ldots, b_n)$ is *3-planar*.

It is easy to see that if $(G, \mathcal{A}, b_1, \ldots, b_n)$ is *3-planar* and G is $(4, \{b_1, \ldots, b_n\})$-connected then $\mathcal{A} = \emptyset$ and $(G, b_1, \ldots, b_n)$ is planar.

We can now state the following result of Seymour [12]; equivalent versions can be found in [1, 13, 14].

Lemma 2.1 *Let G be a graph and s_1, s_2, t_1, t_2 be distinct vertices of G. Then exactly one of the following holds:*

(*i*) *G contains disjoint paths from s_1 to t_1 and from s_2 to t_2.*

(*ii*) (G, s_1, s_2, t_1, t_2) *is 3-planar.*

We also state a generalization of Lemma 2.1, which is a consequence of Theorems 2.3 and 2.4 in [10].

Lemma 2.2 *Let G be a graph, $v_1, \ldots, v_n \in V(G)$ be distinct, and $n \geq 4$. Then exactly one of the following holds:*

(*i*) *There exist $1 \leq i < j < k < l \leq n$ such that G contains disjoint paths from v_i, v_j to v_k, v_l, respectively.*

(*ii*) $(G, v_1, v_2, \ldots, v_n)$ *is 3-planar.*

The next result is Theorem 1.1 in [3].

Lemma 2.3 *Let G be a 5-connected nonplanar graph and let (G_1, G_2) be a 5-separation in G. Suppose $|V(G_i)| \geq 7$ for $i \in [2]$, $a \in V(G_1 \cap G_2)$, and $(G_2 - a, V(G_1 \cap G_2) - \{a\})$ is planar. Then one of the following holds:*

(*i*) *G contains a TK_5 in which a is not a branch vertex.*

(*ii*) *$G - a$ contains K_4^-.*

(*iii*) *G has a 5-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{a, a_1, a_2, a_3, a_4\}$, $G_1 \subseteq G_1'$, and G_2' is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ and the 4-cycle $b_1b_2b_3b_4b_1$ by adding a and the edges ab_i for $i \in [4]$.*

Another result we need is Theorem 1.2 from [3].

Lemma 2.4 *Let G be a 5-connected graph and (G_1, G_2) be a 5-separation in G. Suppose that $|V(G_i)| \geq 7$ for $i \in [2]$ and $G[V(G_1 \cap G_2)]$ contains a triangle aa_1a_2a. Then one of the following holds:*

(*i*) *G contains a TK_5 in which a is not a branch vertex.*

(*ii*) *$G - a$ contains K_4^-.*

(*iii*) *G has a 5-separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{a, a_1, a_2, a_3, a_4\}$ and G_2' is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ and the 4-cycle $b_1b_2b_3b_4b_1$ by adding a and the edges ab_i for $i \in [4]$.*

(*iv*) *For any distinct $u_1, u_2, u_3 \in N(a) - \{a_1, a_2\}$, $G - \{av : v \notin \{a_1, a_2, u_1, u_2, u_3\}\}$ contains TK_5.*

We also need Proposition 4.2 from [3].

Lemma 2.5 *Let G be a 5-connected nonplanar graph and $a \in V(G)$ such that $G - a$ is planar. Then one of the following holds:*

(*i*) *G contains a TK_5 in which a is not a branch vertex.*

(*ii*) $G - a$ *contains* K_4^-.

(*iii*) G *has a* 5*-separation* (G_1, G_2) *such that* $V(G_1 \cap G_2) = \{a, a_1, a_2, a_3, a_4\}$ *and* G_2 *is the graph obtained from the edge-disjoint union of the 8-cycle* $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ *and the 4-cycle* $b_1b_2b_3b_4b_1$ *by adding* a *and the edges* ab_i *for* $i \in [4]$.

We will make use of the following result of Perfect [9] on independent paths. A collection of paths in a graph are said to be *independent* if no internal vertex of a path in this collection belongs to another path in the collection.

Lemma 2.6 *Let* G *be a graph,* $u \in V(G)$*, and* $A \subseteq V(G - u)$*. Suppose there exist* k *independent paths from* u *to distinct* $a_1, \ldots, a_k \in A$*, respectively, and otherwise disjoint from* A*. Then for any* $n \geq k$*, if there exist* n *independent paths* $P_1, \ldots, P_n$ *in* G *from* u *to* n *distinct vertices in* A *and otherwise disjoint from* A *then* $P_1, \ldots, P_n$ *may be chosen so that* $a_i \in V(P_i)$ *for* $i \in [k]$.

We will also use a result of Watkins and Mesner [15] on cycles through three vertices.

Lemma 2.7 *Let* G *be a* 2*-connected graph and let* y_1, y_2, y_3 *be three distinct vertices of* G*. Then there is no cycle in* G *containing* $\{y_1, y_2, y_3\}$ *if, and only if, one of the following statements holds:*

(*i*) *There exists a 2-cut* S *in* G *and there exist pairwise disjoint subgraphs* D_{y_i} *of* $G - S$*,* $i = 1, 2, 3$*, such that* $y_i \in V(D_{y_i})$ *and each* D_{y_i} *is a union of components of* $G - S$.

(*ii*) *There exist 2-cuts* S_{y_i} *of* G*,* $i = 1, 2, 3$*,* $z \in S_{y_1} \cap S_{y_2} \cap S_{y_3}$*, and pairwise disjoint subgraphs* D_{y_i} *of* G*, such that* $y_i \in V(D_{y_i})$*, each* D_{y_i} *is a union of components of* $G - S_{y_i}$*, and* $S_{y_1} - \{z\}, S_{y_2} - \{z\}, S_v - \{z\}$ *are pairwise disjoint.*

(*iii*) *There exist pairwise disjoint* 2*-cuts* S_{y_i} *in* G*,* $i = 1, 2, 3$*, and pairwise disjoint subgraphs* D_{y_i} *of* $R - S_{y_i}$ *such that* $y_i \in V(D_{y_i})$*, each* D_{y_i} *is a union of components of* $G - S_{y_i}$*, and* $G - V(D_{y_1} \cup D_{y_2} \cup D_{y_3})$ *has precisely two components, each containing exactly one vertex from* S_{y_i} *for* $i \in [3]$.

3 Nonseparating paths

Our first step for proving Theorem 1.1 is to find the path X in G (see Figure 1) whose removal does not affect connectivity too much.

We need the concept of chain of blocks. Let G be a graph and $\{u, v\} \subseteq V(G)$. We say that a sequence of blocks $B_1, \ldots, B_k$ in G is a *chain of blocks* from u to v if either $k = 1$ and $u, v \in V(B_1)$ are distinct, or $k \geq 2$, $u \in V(B_1) - V(B_2)$, $v \in V(B_k) - V(B_{k-1})$, $|V(B_i) \cap V(B_{i+1})| = 1$ for $i \in [k-1]$, and $V(B_i) \cap V(B_j) = \emptyset$ for any $i, j \in [k]$ with $|i - j| \geq 2$. For convenience, we also view this chain of blocks as $\bigcup_{i=1}^k B_i$, a subgraph of G.

The following result was implicit in [2,5]. Since it has not been stated and proved explicitly before, we include a proof. We need the concept of a bridge. Let G be a graph and H a subgraph of G. Then an H*-bridge* of G is a subgraph of G that is either induced by an edge of $G - E(H)$ with both ends in $V(H)$, or induced by the edges in some component of $G - H$ as well as those edges of G from that component to H.

Lemma 3.1 *Let G be a graph and let $x_1, x_2, y_1, y_2 \in V(G)$ be distinct such that G is $(4, \{x_1, x_2, y_1, y_2\})$-connected. Suppose there exists a path X in $G - x_1x_2$ from x_1 to x_2 such that $G - X$ contains a chain of blocks B from y_1 to y_2. Then one of the following holds:*

(*i*) *There is a 4-separation (G_1, G_2) in G such that $B + \{x_1, x_2\} \subseteq G_1$, $|V(G_2)| \geq 6$, and $(G_2, V(G_1 \cap G_2))$ is planar.*

(*ii*) *There exists an induced path X' in $G - x_1x_2$ from x_1 to x_2 such that $G - X'$ is a chain of blocks from y_1 to y_2 and contains B.*

Proof. Without loss of generality, we may assume that X is induced in $G - x_1x_2$. We choose such X that

(1) B is maximal,

(2) the smallest size of a component of $G - X$ disjoint from B (if exists) is minimal, and

(3) the number of components of $G - X$ is minimal.

We claim that $G - X$ is connected. For, suppose $G - X$ is not connected and let D be a component of $G - X$ other than B such that $|V(D)|$ is minimal. Let $u, v \in N(D) \cap V(X)$ such that uXv is maximal. Since G is $(4, \{x_1, x_2, y_1, y_2\})$-connected, $uXv - \{u, v\}$ contains a neighbor of some component of $G - X$ other than D. Let Q be an induced path in $G[D + \{u, v\}]$ from u to v, and let X' be obtained from X by replacing uXv with Q. Then B is contained in B', the chain of blocks in $G - X'$ from y_1 to y_2. Moreover, either the smallest size of a component of $G - X'$ disjoint from B' is smaller than the smallest size of a component of $G - X$ disjoint from B, or the number of components of $G - X'$ is smaller than the number of components of $G - X$. This gives a contradiction to (1) or (2) or (3). Hence, $G - X$ is connected.

If $G - X = B$, we are done with $X' := X$. So assume $G - X \neq B$. By (1), each B-bridge of $G - X$ has exactly one vertex in B. Thus, for each B-bridge D of $G - X$, let $b_D \in V(D) \cap V(B)$ and $u_D, v_D \in N(D - b_D) \cap V(X)$ such that $u_D X v_D$ is maximal.

We now define a new graph $\mathcal{B}$ such that $V(\mathcal{B})$ is the set of all B-bridges of $G - X$, and two B-bridges in $G - X$, C and D, are adjacent if $u_C X v_C - \{u_C, v_C\}$ contains a neighbor of $D - b_D$ or $u_D X v_D - \{u_D, v_D\}$ contains a neighbor of $C - b_C$. Let $\mathcal{D}$ be a component of $\mathcal{B}$. Then $\bigcup_{D \in V(\mathcal{D})} u_D X v_D$ is a subpath of X. Let $S_{\mathcal{D}}$ be the union of $\{b_D : D \in V(\mathcal{D})\}$ and the set of neighbors in B of the internal vertices of $\bigcup_{D \in V(\mathcal{D})} u_D X v_D$.

Suppose $\mathcal{B}$ has a component $\mathcal{D}$ such that $|S_{\mathcal{D}}| \leq 2$. Let $u, v \in V(X)$ such that $uXv = \bigcup_{D \in V(\mathcal{D})} u_D X v_D$. Then $\{u, v\} \cup S_{\mathcal{D}}$ is a cut in G. Since G is $(4, \{x_1, x_2, y_1, y_2\})$-connected, $|S_{\mathcal{D}}| = 2$. So there is a 4-separation (G_1, G_2) in G such that $V(G_1 \cap G_2) = \{u, v\} \cup S_{\mathcal{D}}$, $B + \{x_1, x_2\} \subseteq G_1$, and $D \subseteq G_2$ for $D \in V(\mathcal{D})$. Hence $|V(G_2)| \geq 6$. If G_2 has disjoint paths S_1, S_2, with S_1 from u to v and S_2 between the vertices in $S_{\mathcal{D}}$, then choose S_1 to be induced and let $X' = x_1 X u \cup S_1 \cup v X x_2$; now $B \cup S_2$ is contained in the chain of blocks in $G - X'$ from y_1 to y_2, contradicting (1). So no such two paths exist. Hence, by Lemma 2.1, $(G_2, V(G_1 \cap G_2))$ is planar and thus (i) holds.

Therefore, we may assume that $|S_{\mathcal{D}}| \geq 3$ for any component $\mathcal{D}$ of $\mathcal{B}$. Hence, there exist a component $\mathcal{D}$ of $\mathcal{B}$ and $D \in V(\mathcal{D})$ with the following property: $u_D X v_D - \{u_D, v_D\}$ contains vertices w_1, w_2 and $S_{\mathcal{D}}$ contains distinct vertices b_1, b_2 such that for each $i \in [2]$, $\{b_i, w_i\}$ is contained in a $(B \cup X)$-bridge of G disjoint from $D - b_D$. Let P denote an induced path in $G[D + \{u_D, v_D\}]$ between u_D and v_D, and let X' be obtained from X by replacing $u_D X v_D$ with P. Clearly, the chain of blocks in $G - X'$ from y_1 to y_2 contains B as well as a path from b_1 to b_2 and internally disjoint from $D \cup B$. This is a contradiction to (1). ∎

We now show that the conclusion of Theorem 1.1 holds or we can find a path X in G such that $y_1, y_2 \notin V(X)$ and $(G - y_2) - X$ is 2-connected.

Lemma 3.2 *Let G be a 5-connected nonplanar graph and let $x_1, x_2, y_1, y_2 \in V(G)$ be distinct such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ with $y_1y_2 \notin E(G)$. Then one of the following holds:*

(i) *G contains a TK_5 in which y_2 is not a branch vertex.*

(ii) *$G - y_2$ contains K_4^-.*

(iii) *G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{y_2, a_1, a_2, a_3, a_4\}$ and G_2 is the graph obtained from the edge-disjoint union of the 8-cycle $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ and the 4-cycle $b_1b_2b_3b_4b_1$ by adding y_2 and the edges y_2b_i for $i \in [4]$.*

(iv) *For $w_1, w_2, w_3 \in N(y_2) - \{x_1, x_2\}$, $G - \{y_2v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}$ contains TK_5, or $G - x_1x_2$ has an induced path X from x_1 to x_2 such that $y_1, y_2 \notin V(X)$, $w_1, w_2, w_3 \in V(X)$, and $(G - y_2) - X$ is 2-connected.*

Proof. First, we may assume that

(1) $G - x_1x_2$ has an induced path X from x_1 to x_2 such that $y_1, y_2 \notin V(X)$ and $(G - y_2) - X$ is 2-connected.

To see this, let $z \in N(y_1) - \{x_1, x_2\}$. Since G is 5-connected, $(G - x_1x_2) - \{y_1, y_2, z\}$ has a path X from x_1 to x_2. Thus, we may apply Lemma 3.1 to $G - y_2$, X and $B = y_1z$.

Suppose (i) of Lemma 3.1 holds. Then G has a 5-separation (G_1, G_2) such that $y_2 \in V(G_1 \cap G_2)$, $\{x_1, x_2, y_1, z\} \subseteq V(G_1)$ and $y_1z \in E(G_1)$, $|V(G_2)| \geq 7$, and $(G_2 - y_2, V(G_1 \cap G_2) - \{y_2\})$ is planar. If $|V(G_1)| \geq 7$ then, by Lemma 2.3, (i) or (ii) or (iii) holds. If $|V(G_1)| = 5$ then $G_1 - y_2$ has a K_4^- or $G - y_2$ is planar; hence, (ii) holds in the former case, and (i) or (ii) or (iii) holds in the latter case by Lemma 2.5. Thus we may assume that $|V(G_1)| = 6$. Let $v \in V(G_1 - G_2)$. Then $v \neq y_2$. Since G is 5-connected, v must be adjacent to all vertices in $V(G_1 \cap G_2)$. Thus, $v \neq y_1$ as $y_1y_2 \notin E(G)$. Now $|V(G_1 \cap G_2) \cap \{x_1, x_2, z\}| \geq 2$. Therefore, $G[\{v, y_1\} \cup (V(G_1 \cap G_2) \cap \{x_1, x_2, z\})]$ contains K_4^-; so (ii) holds.

So we may assume that (ii) of Lemma 3.1 holds. Then $(G - y_2) - x_1x_2$ has an induced path, also denoted by X, from x_1 to x_2 such that $(G - y_2) - X$ is a chain of blocks from y_1 to z. Since $zy_1 \in E(G)$, $(G - y_2) - X$ is in fact a block. If $V((G - y_2) - X) = \{y_1, z\}$ then, since G is 5-connected and X is induced in $(G - y_2) - x_1x_2$, $G[\{x_1, x_2, z, y_1\}] \cong K_4$; so (ii) holds. This completes the proof of (1).

We wish to prove (iv). So let $w_1, w_2, w_3 \in N(y_2) - \{x_1, x_2\}$ and assume that

$$G' := G - \{y_2v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}$$

does not contain TK_5. We may assume that

(2) $w_1, w_2, w_3 \notin V(X)$.

For, suppose not. If $w_1, w_2, w_3 \in V(X)$ then (iv) holds. So, without loss of generality, we may assume $w_1 \in V(X) - \{x_1, x_2\}$ and $w_2 \in V(G - X)$. Since X is induced in $G - x_1x_2$ and G is 5-connected, $(G - y_2) - (X - w_1)$ is 2-connected and, hence, contains independent paths P_1, P_2 from y_1 to w_1, w_2, respectively. Then $w_1Xx_1 \cup w_1Xx_2 \cup w_1y_2 \cup P_1 \cup (y_2w_2 \cup P_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_1, y_2, a contradiction.

(3) For any $u \in V(x_1Xx_2) - \{x_1, x_2\}$, $\{u, y_1, y_2\}$ is not contained in any cycle in $G' - (X - u)$.

For, suppose there exists $u \in V(x_1Xx_2) - \{x_1, x_2\}$ such that $\{u, y_1, y_2\}$ is contained in a cycle C in $G' - (X - u)$. Then $uXx_1 \cup uXx_2 \cup C \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices u, x_1, x_2, y_1, y_2, a contradiction. So we have (3).

Let $y_3 \in V(X)$ such that $y_3x_2 \in E(X)$, and let $H := G' - (X - y_3)$. Note that H is 2-connected. By (3), no cycle in H contains $\{y_1, y_2, y_3\}$. Thus, we apply Lemma 2.7 to H. In order to treat simultaneously the three cases in the conclusion of Lemma 2.7, we introduce some notation. Let $S_{y_i} = \{a_i, b_i\}$ for $i \in [3]$, such that if Lemma 2.7(i) occurs we let $a_1 = a_2 = a_3$, $b_1 = b_2 = b_3$, and $S_{y_i} = S$ for $i \in [3]$; if Lemma 2.7(ii) occurs then $a_1 = a_2 = a_3$; and if Lemma 2.7(iii) then $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ belong to different components of $H - V(D_{y_1} \cup D_{y_2} \cup D_{y_3})$. If Lemma 2.7$(ii)$ or Lemma 2.7(iii) occurs then let B_a, B_b denote the components of $H - V(D_{y_1} \cup D_{y_2} \cup D_{y_3})$ such that for $i \in [3]$ $a_i \in V(B_a)$ and $b_i \in V(B_b)$. Note that $B_a = B_b$ is possible, but only if Lemma 2.7(ii) occurs.

For convenience, let $D'_i := G'[D_{y_i} + \{a_i, b_i\}]$ for $i \in [3]$. We choose the cuts S_{y_i} so that

(4) $D'_1 \cup D'_2 \cup D'_3$ is maximal.

Since H is 2-connected, D'_i, for each $i \in [3]$, contains a path Y_i from a_i to b_i and through y_i. In addition, since $(G - y_2) - X$ is 2-connected, for any $v \in V(D'_3) - \{a_3, b_3, y_3\}$, $D'_3 - y_3$ contains a path from a_3 to b_3 through v.

(5) If $B_a \cap B_b = \emptyset$ then $|V(B_a)| = 1$ or B_a is 2-connected, and $|V(B_b)| = 1$ or B_b is 2-connected. If $B_a \cap B_b \neq \emptyset$ then $B_a = B_b$ and $B_a - a_3$ is 2-connected.

First, suppose $B_a \cap B_b = \emptyset$. By symmetry, we only prove the claim for B_a. Suppose $|V(B_a)| > 1$ and B_a is not 2-connected. Then B_a has a separation (B_1, B_2) such that $|V(B_1 \cap B_2)| \leq 1$. Since H is 2-connected, $|V(B_1 \cap B_2)| = 1$ and, for some permutation ijk of [3], $a_i \in V(B_1) - V(B_2)$ and $a_j, a_k \in V(B_2)$. Replacing S_{y_i}, D'_i by $V(B_1 \cap B_2) \cup \{b_i\}, D'_i \cup B_1$, respectively, while keeping $S_{y_j}, D'_j, S_{y_k}, D'_k$ unchanged, we derive a contradiction to (4).

Now assume $B_a \cap B_b \neq \emptyset$. Then $B_a = B_b$ by definition, and $a_1 = a_2 = a_3$ by our assumption above. Suppose $B_a - a_3$ is not 2-connected. Then B_a has a 2-separation (B_1, B_2) with $a_3 \in V(B_1 \cap B_2)$. First, suppose for some permutation ijk of [3], $b_i \in V(B_1) - V(B_2)$ and $b_j, b_k \in V(B_2)$. Then replacing S_{y_i}, D'_i by $V(B_1 \cap B_2), D'_i \cup B_1$, respectively, while keeping $S_{y_j}, D'_j, S_{y_k}, D'_k$ unchanged, we derive a contradiction to (4). Therefore, we may assume $\{b_1, b_2, b_3\} \subseteq V(B_1)$. Since G is 5-connected, there exists $rr' \in E(G)$ such that $r \in V(X) - \{y_3, x_2\}$ and $r' \in V(B_2 - B_1)$. Let R be a path $B_2 - (B_1 - a_3)$ from a_3 to r', and R' a path in $B_1 - B_2$ from b_1 to b_2. Then $(R \cup r'r \cup rXx_1) \cup (a_3Y_3y_3 \cup y_3x_2) \cup a_3Y_1y_1 \cup a_3Y_2y_2 \cup (y_1Y_1b_1 \cup R' \cup b_2Y_2y_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices a_3, x_1, x_2, y_1, y_2, a contradiction.

(6) D_{y_i} is connected for $i \in [3]$.

Suppose D_{y_i} is not connected for some $i \in [3]$, and let D be a component of D_{y_i} not containing y_i. Since G is 5-connected, there exists $rr' \in E(G)$ such that $r \in V(X) - \{x_2, y_3\}$ and $r' \in V(D)$.

Let R be a path in $G[D + a_i]$ from a_i to r', and R' a path from b_1 to b_2 in $B_b - a_3$. By (5), let A_1, A_2, A_3 be independent paths in B_a from a_i to a_1, a_2, a_3, respectively. Then $(R \cup r'r \cup rXx_1) \cup (A_1 \cup a_1Y_1y_1) \cup (A_2 \cup a_2Y_2y_2) \cup (A_3 \cup a_3Y_3y_3 \cup y_3x_2) \cup (y_1Y_1b_1 \cup R' \cup b_2Y_2y_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices a_i, x_1, x_2, y_1, y_2, a contradiction.

(7) If $a_1 = a_2 = a_3$ then $N(a_3) \cap V(X - \{x_2, y_3\}) = \emptyset$.

For, suppose $a_1 = a_2 = a_3$ and there exists $u \in N(a_3) \cap V(X - \{x_2, y_3\})$. Let Q be a path in $B_b - a_3$ between b_1 and b_2, and let P be a path in $D'_3 - b_3$ from a_3 to y_3. Then $(a_3u \cup uXx_1) \cup (P \cup y_3x_2) \cup a_3Y_1y_1 \cup a_3Y_2y_2 \cup (y_1Y_1b_1 \cup Q \cup b_2Y_2y_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices a_3, x_1, x_2, y_1, y_2, a contradiction.

We may assume that

(8) there exists $u \in V(X) - \{x_1, x_2, y_3\}$ such that $N(u) - \{y_2\} \not\subseteq V(X \cup D'_3)$.

For, suppose no such vertex exists. Then G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a_3, b_3, x_1, x_2, y_2\}$, $X \cup D'_3 \subseteq G_1$, and $D'_1 \cup D'_2 \cup B_a \cup B_b \subseteq G_2$. Clearly, $|V(G_2)| \geq 7$ since $|N(y_1)| \geq 5$ and $y_1y_2 \notin E(G)$. If $|V(G_1)| \geq 7$ then, by Lemma 2.4, (i) or (ii) or (iii) or (iv) holds. So we may assume $|V(G_1)| = 6$. Then $X = x_1y_3x_2$ and $V(D_{y_3}) = \{y_3\}$. Hence, $G[\{x_1, x_2, y_1, y_3\}] \cong K_4^-$; so (ii) holds.

(9) For all $u \in V(X) - \{x_1, x_2, y_3\}$ with $N(u) - \{y_2\} \not\subseteq V(X \cup D'_3)$, $N(u) \cap V(D'_3 - y_3) = \emptyset$.

For, suppose there exist $u \in V(X) - \{x_1, x_2, y_3\}$, $u_1 \in (N(u) - \{y_2\}) - V(X \cup D'_3)$, and $u_2 \in N(u) \cap V(D'_3 - y_3)$. Recall (see before (5)) that there is a path Y'_3 in $D'_3 - y_3$ from a_3 to b_3 through u_2.

Suppose $u_1 \in V(D_{y_i})$ for some $i \in [2]$. Then $D'_i - b_i$ (or $D'_i - a_i$) has a path Y'_i from u_1 to a_i (or b_i) through y_i. If Y'_i ends at a_i then let P_a, P_b be disjoint paths in $B_a \cup B_b$ from a_1, b_3 to a_2, b_{3-i}, respectively; now $Y'_i \cup P_a \cup Y_{3-i} \cup P_b \cup b_3Y'_3u_2 \cup u_2uu_1$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3). So Y'_i ends at b_i. Let P_b, P_a be disjoint paths in $B_a \cup B_b$ from b_1, a_{3-i} to b_2, a_3, respectively. Then $Y'_i \cup P_b \cup Y_{3-i} \cup P_a \cup a_3Y'_3u_2 \cup u_2uu_1$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3).

Thus, $u_1 \in V(B_a \cup B_b)$. By symmetry and (7), assume $u_1 \in V(B_b)$. Note that $u_1 \notin \{a_3, b_3\}$ (by the choice of u_1) and $B_b - a_3$ is 2-connected (by (5)). Hence, $B_b - a_3$ has disjoint paths Q_1, Q_2 from $\{u_1, b_3\}$ to $\{b_1, b_2\}$. By symmetry between b_1 and b_2, we may assume Q_1 is between u_1 and b_1 and Q_2 is between b_3 and b_2. Let P be a path in B_a from a_1 to a_2 (which is trivial if $|V(B_a)| = 1$). Then $Q_1 \cup u_1uu_2 \cup u_2Y'_3b_3 \cup Q_2 \cup Y_2 \cup P \cup Y_1$ is a cycle in $G' - (X - u)$ containing $\{y_1, y_2, u\}$, contradicting (3).

(10) For any $u \in V(X) - \{x_1, x_2, y_3\}$ with $N(u) - \{y_2\} \not\subseteq V(X \cup D'_3)$, there exists $i \in [2]$ such that $N(u) - \{y_2\} \subseteq V(D'_i)$ and $\{a_i, b_i\} \not\subseteq N(u)$.

To see this, let $u_1, u_2 \in (N(u) - \{y_2\}) - V(X \cup D_3')$ be distinct, which exist by (9) (and since X is induced in $G' - x_1x_2$). Suppose we may choose such u_1, u_2 so that $\{u_1, u_2\} \not\subseteq V(D_i')$ for $i \in [2]$.

We claim that $\{u_1, u_2\} \not\subseteq V(B_a)$ and $\{u_1, u_2\} \not\subseteq V(B_b)$. Recall that if $B_a \cap B_b \neq \emptyset$ then $B_a = B_b$ and if $B_a \cap B_b = \emptyset$ then there is symmetry between B_a and B_b. So if the claim fails we may assume that $u_1, u_2 \in V(B_b)$. Then by (5), $B_b - a_3$ is 2-connected; so $B_b - a_3$ contains disjoint paths Q_1, Q_2 from $\{u_1, u_2\}$ to $\{b_1, b_2\}$. If $B_a = B_b$, let $P = a_3$. If $B_a \cap B_b = \emptyset$, then let P be a path in B_a from a_1 to a_2. Now $Q_1 \cup u_1uu_2 \cup Q_2 \cup Y_1 \cup P \cup Y_2$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3).

Next, we show that $\{a_i, b_i\} \not\subseteq N(u)$ for $i \in [2]$. For, suppose $u_1 = a_i$ and $u_2 = b_i$ for some $i \in [2]$. Then, since $\{u_1, u_2\} \cap \{a_3, b_3\} = \emptyset$, $|V(B_a)| \geq 2$ and $|V(B_b)| \geq 2$. By (5), let P_1, P_2 be independent paths in B_a from a_i to a_{3-i}, a_3, respectively, and Q_1, Q_2 be independent paths in B_b from b_i to b_{3-i}, b_3, respectively. Now $ua_i \cup ub_i \cup a_iY_iy_i \cup b_iY_iy_i \cup (y_ix_1 \cup x_1Xu) \cup (P_1 \cup Y_{3-i} \cup Q_1) \cup (P_2 \cup a_3Y_3y_3) \cup (Q_2 \cup b_3Y_3y_3) \cup uXy_3 \cup y_ix_2y_3$ is a TK_5 in G' with branch vertices a_i, b_i, u, y_i, y_3, a contradiction.

Suppose $u_1 \in V(B_a - a_3)$ and $u_2 \in V(B_b - b_3)$. Then $|V(B_a)| \geq 2$ and $|V(B_b)| \geq 2$. Let Y_3' be a path in $D_3' - y_3$ from a_3 to b_3. First, assume that $u_1 \in \{a_1, a_2\}$ or $u_2 \in \{b_1, b_2\}$. By symmetry, we may assume $u_1 = a_1$. So $u_2 \neq b_1$. By (5), $B_a - a_1$ contains a path P from a_2 to a_3, and B_b contains disjoint paths Q_1, Q_2 from $\{b_2, b_3\}$ to b_1, u_2, respectively. Then $Y_1 \cup Q_1 \cup Y_2 \cup P \cup Y_3' \cup Q_2 \cup u_1uu_2$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3). So $u_1 \notin \{a_1, a_2\}$ and $u_2 \notin \{b_1, b_2\}$. Then by (5) and symmetry, we may assume that B_a contains disjoint paths P_1, P_2 from u_1, a_3 to a_1, a_2, respectively. By (5) again, B_b contains disjoint paths Q_1, Q_2 from b_1, u_2, respectively to $\{b_2, b_3\}$. Now $P_1 \cup Y_1 \cup Q_1 \cup Y_2 \cup P_2 \cup Y_3' \cup Q_2 \cup u_2uu_1$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3).

Therefore, we may assume $u_1 \in V(D_{y_i})$ for some $i \in [2]$. By symmetry, we may assume that $u_1 \in V(D_{y_1})$ and $D_1' - a_1$ contains a path R_1 from u_1 to b_1 and through y_1. Then $u_2 \notin V(D_1')$ as we assumed $\{u_1, u_2\} \not\subseteq V(D_1')$.

Suppose $u_2 \in V(D_{y_2})$. If $D_2' - a_2$ contains a path R_2 from u_2 to b_2 through y_2 then let Q be a path in B_b from b_1 to b_2; now $R_1 \cup Q \cup R_2 \cup u_2uu_1$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3). So $D_2' - b_2$ contains a path R_2 from u_2 to a_2 and through y_2. Now let P be a path in B_a from a_2 to a_3, Q be a path in $B_b - a_3$ from b_1 to b_3. Let Y_3' be a path in $D_3' - y_3$ from a_3 to b_3. Then $R_1 \cup Q \cup Y_3' \cup P \cup R_2 \cup u_2uu_1$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3).

Finally, assume $u_2 \in V(B_a \cup B_b)$. If $u_2 \in V(B_b)$ then, by (5), let Q_1, Q_2 be disjoint paths in $B_b - a_3$ from b_1, u_2, respectively, to $\{b_2, b_3\}$, and let P be a path in B_a from a_2 to a_3; now $u_2uu_1 \cup R_1 \cup Q_1 \cup Q_2 \cup Y_2 \cup P \cup Y_3'$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3). So $u_2 \notin V(B_b)$ and $u_2 \in V(B_a - a_1)$; hence $B_a \cap B_b = \emptyset$. Let P be a path in B_a from u_2 to a_2 and Q be a path in B_b from b_1 to b_2. Then $u_2uu_1 \cup R_1 \cup Q \cup Y_2 \cup P$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3). This completes the proof of (10).

By (10) and by symmetry, let $u \in V(X) - \{x_1, x_2, y_3\}$ and $u_1, u_2 \in N(u)$ such that $u_1 \in V(D_{y_1})$ and $u_2 \in V(D_1')$. If $G[D_1' + u]$ contains independent paths R_1, R_2 from u to a_1, b_1, respectively, such that $y_1 \in V(R_1 \cup R_2)$, then let P be a path in B_a between a_1 and a_2 and Q be a path in $B_b - a_3$ between b_1 and b_2; now $R_1 \cup P \cup Y_2 \cup Q \cup R_2$ is a cycle in $G' - (X - u)$ containing $\{u, y_1, y_2\}$, contradicting (3). So such paths do not exist. Then in the 2-connected graph $D_1^* := G[D_1' + u] + \{c, ca_1, cb_1\}$ (by adding a new vertex c), there is no cycle containing $\{c, u, y_1\}$. Hence, by Lemma 2.7, D_1^* has a 2-cut T separating y_1 from $\{u, c\}$, and $T \cap \{u, c\} = \emptyset$.

We choose u, u_1, u_2 and T so that the T-bridge of D_1^* containing y_1, denoted B, is minimal. Then $B - T$ contains no neighbor of $X - \{x_1, x_2\}$. Hence, G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{x_1, x_2, y_2\} \cup V(T)$, $B \subseteq G_1$, and $X \cup D_2' \cup D_3' \subseteq G_2$. Clearly, $|V(G_2)| \geq 7$. Since $y_1 y_2 \notin E(G)$ and G is 5-connected, $|V(G_1)| \geq 7$. So (i) or (ii) or (iii) or (iv) holds by Lemma 2.4. ∎

4 An intermediate substructure

By Lemma 3.2, to prove Theorem 1.1 it suffices to deal with the second part of (iv) of Lemma 3.2. Thus, let G be a 5-connected nonplanar graph and $x_1, x_2, y_1, y_2 \in V(G)$ be distinct such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ with $y_1 y_2 \notin E(G)$, let $w_1, w_2, w_3 \in N(y_2) - \{x_1, x_2\}$ be distinct, and let P be an induced path in $G - x_1 x_2$ from x_1 to x_2 such that $y_1, y_2 \notin V(P)$, $w_1, w_2, w_3 \in V(P)$, and $(G - y_2) - P$ is 2-connected.

Without loss of generality, assume x_1, w_1, w_2, w_3, x_2 occur on P in order. Let

$$X := x_1 P w_1 \cup w_1 y_2 w_3 \cup w_3 P x_2,$$

and let

$$G' := G - \{y_2 v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}.$$

Then X is an induced path in $G' - x_1 x_2$, $y_1 \notin V(X)$, and $G' - X$ is 2-connected. For convenience, we record this situation by calling $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3)$ a 9-*tuple*.

In this section, we obtain a substructure of G' in terms of X and seven additional paths A, B, C, P, Q, Y, Z in G'. See Figure 1, where X is the path in boldface and Y, Z are not shown. First, we find two special paths Y, Z in G' with Lemma 4.1 below. We will then use Lemma 4.2 to find the paths A, B, C, and use Lemma 4.3 to find the paths P and Q. In the next section, we will use this substructure to find the desired TK_5 in G or G'.

Lemma 4.1 *Let $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3)$ be a 9-tuple. Then one of the following holds:*

(i) G contains TK_5 in which y_2 is not a branch vertex, or G' contains TK_5.

(ii) $G - y_2$ contains K_4^-.

(iii) G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{y_2, a_1, a_2, a_3, a_4\}$, G_2 is the graph obtained from the edge-disjoint union of the 8-cycle $a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4 a_1$ and the 4-cycle $b_1 b_2 b_3 b_4 b_1$ by adding y_2 and the edges $y_2 b_i$ for $i \in [4]$.

(iv) There exist $z_1 \in V(x_1 X y_2) - \{x_1, y_2\}$, $z_2 \in V(x_2 X y_2) - \{x_2, y_2\}$ such that $H := G' - (V(X - \{y_2, z_1, z_2\}) \cup E(X))$ has disjoint paths Y, Z from y_1, z_1 to y_2, z_2, respectively.

Proof. Let K be the graph obtained from $G - \{x_1, x_2, y_2\}$ by contracting $x_i X y_2 - \{x_i, y_2\}$ to the new vertex u_i, for $i \in [2]$. Note that K is 2-connected; since G is 5-connected, X is induced in $G' - x_1 x_2$, and $G - X$ is 2-connected. We may assume that

(1) there exists a collection $\mathcal{A}$ of subsets of $V(K)-\{u_1, u_2, w_2, y_1\}$ such that $(K, \mathcal{A}, u_1, y_1, u_2, w_2)$ is 3-planar.

For, suppose this is not the case. Then by Lemma 2.1, K contains disjoint paths, say Y, U, from y_1, u_1 to w_2, u_2, respectively. Let v_i denote the neighbor of u_i in the path U, and let $z_i \in V(x_i X y_2)-\{x_i, y_2\}$ be a neighbor of v_i in G. Then $Z := (U-\{u_1, u_2\})+\{z_1, z_2, z_1v_1, z_2v_2\}$ is a path between z_1 and z_2. Now $Y + \{y_2, y_2w_2\}, Z$ are the desired paths for (iv). So we may assume (1).

Since $G - X$ is 2-connected, $|N_K(A) \cap \{u_1, u_2, w_2\}| \leq 1$ for all $A \in \mathcal{A}$. Let $p(K, \mathcal{A})$ be the graph obtained from K by (for each $A \in \mathcal{A}$) deleting A and adding new edges joining every pair of distinct vertices in $N_K(A)$. Since G is 5-connected and $G - X$ is 2-connected, we may assume that $p(K, \mathcal{A}) - \{u_1, u_2\}$ is a 2-connected plane graph, and for each $A \in \mathcal{A}$ with $N_K(A) \cap \{u_1, u_2\} \neq \emptyset$ the edge joining vertices of $N_K(A) - \{u_1, u_2\}$ occur on the outer cycle D of $p(K, \mathcal{A}) - \{u_1, u_2\}$. Note that $y_1, w_2 \in V(D)$.

Let $t_1 \in V(D)$ with t_1Dy_1 minimal such that $u_1t_1 \in E(p(K, \mathcal{A}))$; and let $t_2 \in V(D)$ with y_1Dt_2 minimal such that $u_2t_2 \in E(p(K, \mathcal{A}))$. (So t_1, y_1, t_2, w_2 occur on D in clockwise order.) Since K is 2-connected and X is induced in $G'-x_1x_2$, there exist $z_1 \in V(x_1Xy_2)-\{x_1, y_2\}$ and independent paths R_1, R_1' in G from z_1 to D and internally disjoint from $V(p(K, \mathcal{A})) \cup V(X)$, such that R_1 ends at t_1 and R_1' ends at some vertex $t_1' \neq t_1$, and w_2, t_1', t_1, y_1 occur on D in clockwise order. Similarly, there exist $z_2 \in V(x_2Xy_2)-\{x_2, y_2\}$ and independent paths R_2, R_2' in G from z_2 to D and internally disjoint from $V(p(K, \mathcal{A})) \cup V(X)$, such that R_2 ends at t_2, R_2' ends at some vertex $t_2' \neq t_2$, and y_1, t_2, t_2', w_2 occur on D in clockwise order.

We may assume that

(2) $K-\{u_1, u_2\}$ has no 2-separation (K', K'') such that $V(K'\cap K'') \subseteq V(t_1Dt_2)$, $|V(K')| \geq 3$, and $V(t_2Dt_1) \subseteq V(K'')$.

For, suppose such a separation (K', K'') does exist in $K - \{u_1, u_2\}$. Then by the definition of u_1, u_2, we see that G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = V(K' \cap K'') \cup \{x_1, x_2, y_2\}$, $K' \subseteq V(G_1)$ and $K'' \cup X \subseteq G_2$. Note that $G[\{x_1, x_2, y_2\}]$ is a triangle in G, $|V(G_2)| \geq 7$, and $|V(G_1)| \geq 6$ (as $|V(K')| \geq 3$). If $|V(G_1)| \geq 7$ then by Lemma 2.4, (i) or (ii) or (iii) holds. (Note that if (iv) of Lemma 2.4 holds then G' has a TK_5; so (i) holds.) So assume $|V(G_1)| = 6$, and let $v \in V(G_1 - G_2)$. Since G is 5-connected, $N(v) = V(G_1 \cap G_2)$. In particular, $v \neq y_1$ as $y_1y_2 \notin E(G)$. Then $G[\{v, x_1, x_2, y_1\}]$ contains K_4^-, and (ii) holds. So we may assume (2).

Next we may assume that

(3) each neighbor of x_1 is contained in $V(X)$, or $V(t_1Dy_1)$, or some $A \in \mathcal{A}$ with $u_1 \in N_K(A)$, and each neighbor of x_2 is contained $V(X)$, or $V(y_1Dt_2)$, or some $A \in \mathcal{A}$ with $u_2 \in N_K(A)$.

For, otherwise, we may assume by symmetry that there exists $a \in N(x_1) - V(X)$ such that $a \notin V(t_1Dy_1)$ and $a \notin A$ for $A \in \mathcal{A}$ with $u_1 \in N_K(A)$. Let $a' = a$ and $S = a$ if $a \notin A$ for all $A \in \mathcal{A}$. When $a \in A$ for some $A \in \mathcal{A}$ then by (2), there exists $a' \in N_K(A) - V(t_1Dt_2)$ and let S be a path in $G[A + a']$ from a to a'. By (2) again, there is a path T from a' to some $u \in V(t_2Dt_1)-\{t_1, t_2\}$ in $p(K, \mathcal{A})-\{u_1, u_2, y_2\}-t_1Dt_2$. Then $t_1Dt_2 \cup R_1 \cup R_2$ and $R_2' \cup t_2'Du \cup T$ give independent paths T_1, T_2, T_3 in $G-(X-\{z_1, z_2\})$ with T_1, T_2 from y_1 to z_1, z_2, respectively, and T_3 from a' to z_2. Hence, $z_2Xx_2 \cup z_2Xy_2 \cup T_2 \cup (T_3 \cup S \cup ax_1) \cup (T_1 \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2; so (i) holds.

Label the vertices of w_2Dy_1 and x_1Xy_2 such that $w_2Dy_1 = v_1 \dots v_k$ and $x_1Xy_2 = v_{k+1} \dots v_n$, with $v_1 = w_2$, $v_k = y_1$, $v_{k+1} = x_1$ and $v_n = y_2$. Let G_1 denote the union of x_1Xy_2, $\{v_1, \dots, v_k\}$, $G[A \cup (N_K(A) - u_1)]$ for $A \in \mathcal{A}$ with $u_1 \in N_K(A)$, all edges of G' from x_1Xy_2 to $\{v_1, \dots, v_k\}$, and all edges of G' from x_1Xy_2 to A for $A \in \mathcal{A}$ with $u_1 \in N_K(A)$. Note that G_1 is $(4, \{v_1, \dots, v_n\})$-connected. Similarly, let $y_1Dw_2 = z_1 \dots z_l$ and $x_2Xy_2 = z_{l+1} \dots z_m$, with $z_1 = w_2$, $z_l = y_1$, $z_{l+1} = x_2$ and $z_m = y_2$. Let G_2 denote the union of y_2Xx_2, $\{z_1, \dots, z_l\}$, $G[A \cup (N_K(A) - u_2)]$ for $A \in \mathcal{A}$ with $u_2 \in N_K(A)$, all edges of G' from y_2Xx_2 to $\{z_1, \dots, z_l\}$, and all edges of G' from y_2Xx_2 to A for $A \in \mathcal{A}$ with $u_2 \in N_K(A)$. Note that G_2 is $(4, \{z_1, \dots, z_m\})$-connected.

If both $(G_1, v_1, \dots, v_n)$ and $(G_2, z_1, \dots, z_m)$ are planar then $G - y_2$ is planar; so (i) or (ii) or (iii) holds by Lemma 2.5. Hence, we may assume by symmetry that $(G_1, v_1, \dots, v_n)$ is not planar. Then by Lemma 2.2, there exist $1 \le q < r < s < t \le n$ such that G_1 has disjoint paths Q_1, Q_2 from v_q, v_r to v_s, v_t, respectively, and internally disjoint from $\{v_1, \dots, v_n\}$.

Since (K, u_1, y_1, u_2, w_2) is 3-planar, it follows from the definition of G_1 that $q, r \le k$ and $s, t \ge k+1$. Note that the paths y_1Dt_2, $t_2'Dv_q$, v_rDy_1 give rise to independent paths P_1, P_2, P_3 in $K - \{u_1, u_2\}$, with P_1 from y_1 to t_2, P_2 from t_2' to v_q, and P_3 from v_r to y_1. Therefore, $z_2Xx_2 \cup z_2Xy_2 \cup (R_2 \cup P_1) \cup (R_2' \cup P_2 \cup Q_1 \cup v_sXx_1) \cup (P_3 \cup Q_2 \cup v_tXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. So (i) holds. ∎

Conclusion (iv) of Lemma 4.1 motivates the concept of 11-tuple. We say that $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3, z_1, z_2)$ is an *11-tuple* if

- $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3)$ is a 9-tuple, and $z_i \in V(x_iXy_2) - \{x_i, y_2\}$ for $i \in [2]$,
- $H := G' - (V(X - \{y_2, z_1, z_2\}) \cup E(X))$ contains disjoint paths Y, Z from y_1, z_1 to y_2, z_2, respectively, and
- subject to the above conditions, z_1Xz_2 is maximal.

Since G is 5-connected and X is induced in $G' - x_1x_2$, each z_i ($i \in [2]$) has at least two neighbors in $H - \{y_2, z_1, z_2\}$ (which is 2-connected). Note that y_2 has exactly one neighbor $H - \{y_2, z_1, z_2\}$, namely, w_2. So $H - y_2$ is 2-connected.

Lemma 4.2 *Let $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3, z_1, z_2)$ be an 11-tuple and Y, Z be disjoint paths in $H := G' - (V(X - \{y_2, z_1, z_2\}) \cup E(X))$ from y_1, z_1 to y_2, z_2, respectively. Then G contains a TK_5 in which y_2 is not a branch vertex, or G' contains TK_5, or*

(i) for $i \in [2]$, H has no path through z_i, z_{3-i}, y_1, y_2 in order (so $y_1z_i \notin E(G)$), and

(ii) there exists $i \in [2]$ such that H contains independent paths A, B, C, with A and C from z_i to y_1, and B from y_2 to z_{3-i}.

Proof. First, suppose, for some $i \in [2]$, there is a path P in H from z_i to y_2 such that z_i, z_{3-i}, y_1, y_2 occur on P in order. Then $z_{3-i}Xx_{3-i} \cup z_{3-i}Xy_2 \cup (z_{3-i}Pz_i \cup z_iXx_i) \cup z_{3-i}Py_1 \cup y_1Py_2 \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 with branch vertices $x_1, x_2, y_1, y_2, z_{3-i}$. So we may assume that such P does not exist. Hence by the existence of Y, Z in H, we have $y_1z_1, y_1z_2 \notin E(G)$, and (i) holds.

So from now on we may assume that (i) holds. For each $i \in [2]$, let H_i denote the graph obtained from H by duplicating z_i and y_1, and let z_i' and y_1' denote the duplicates of z_i and y_1, respectively. So in H_i, y_1 and y_1' are not adjacent, and have the same set of neighbors, namely $N_H(y_1)$; and the same holds for z_i and z_i'.

First, suppose for some $i \in [2]$, H_i contains pairwise disjoint paths A', B', C' from $\{z_i, z_i', y_2\}$ to $\{y_1, y_1', z_{3-i}\}$, with $z_i \in V(A'), z_i' \in V(C')$ and $y_2 \in V(B')$. If $z_{3-i} \notin V(B')$, then after identifying y_1 with y_1' and z_i with z_i', we obtain from $A' \cup B' \cup C'$ a path in H from z_{3-i} to y_2 through z_i, y_1 in order, contradicting our assumption that (i) holds. Hence $z_{3-i} \in V(B')$. Then we get the desired paths for (ii) from $A' \cup B' \cup C'$ by identifying y_1 with y_1' and z_i with z_i'.

So we may assume that for each $i \in [2]$, H_i does not contain three pairwise disjoint paths from $\{y_2, z_i, z_i'\}$ to $\{y_1, y_1', z_{3-i}\}$. Then H_i has a separation (H_i', H_i'') such that $|V(H_i' \cap H_i'')| = 2$, $\{y_2, z_i, z_i'\} \subseteq V(H_i')$ and $\{y_1, y_1', z_{3-i}\} \subseteq V(H_i'')$.

We claim that $y_1, y_2, y_1', z_i', z_1, z_2 \notin V(H_i' \cap H_i'')$ for $i \in [2]$. Note that $\{y_1, y_1'\} \neq V(H_i' \cap H_i'')$, since otherwise y_1 would be a cut vertex in H separating z_{3-i} from $\{y_2, z_i\}$. Now suppose one of y_1, y_1' is in $V(H_i' \cap H_i'')$; then since y_1, y_1' are duplicates, the vertex in $V(H_i' \cap H_i'') - \{y_1, y_1'\}$ is a cut vertex in H separating $\{y_1, z_{3-i}\}$ from $\{y_2, z_i\}$, a contradiction. So $y_1, y_1' \notin V(H_i' \cap H_i'')$. Similar argument shows that $z_i, z_i' \notin V(H_i' \cap H_i'')$. Since $H - y_2$ is 2-connected, $y_2 \notin V(H_i' \cap H_i'')$. Since $H - \{z_{3-i}, y_2\}$ is 2-connected, $z_{3-i} \notin V(H_i' \cap H_i'')$.

For $i \in [2]$, let $V(H_i' \cap H_i'') = \{s_i, t_i\}$, and let F_i' (respectively, F_i'') be obtained from H_i' (respectively, H_i'') by identifying z_i' with z_i (respectively, y_1' with y_1). Then (F_i', F_i'') is a 2-separation in H such that $V(F_i' \cap F_i'') = \{s_i, t_i\}$, $\{y_2, z_i\} \subseteq V(F_i') - \{s_i, t_i\}$, and $\{y_1, z_{3-i}\} \subseteq V(F_i'') - \{s_i, t_i\}$. Let Z_1, Y_2 denote the $\{s_1, t_1\}$-bridges of F_1' containing z_1, y_2, respectively; and let Z_2, Y_1 denote the $\{s_1, t_1\}$-bridges of F_1'' containing z_2, y_1, respectively.

We may assume $Y_1 = Z_2$ or $Y_2 = Z_1$. For, suppose $Y_1 \neq Z_2$ and $Y_2 \neq Z_1$. Since $H - y_2$ is 2-connected, there exist independent P_1, Q_1 in Z_1 from z_1 to s_1, t_1, respectively, independent paths P_2, Q_2 in Z_2 from z_2 to s_1, t_1, respectively, independent paths P_3, Q_3 in Y_1 from y_1 to s_1, t_1, respectively, and a path S in Y_2 from y_2 to one of $\{s_1, t_1\}$ and avoiding the other, say avoiding t_1. Then $z_1Xx_1 \cup z_1Xy_2 \cup y_2x_1 \cup P_1 \cup S \cup (P_3 \cup y_1x_1) \cup (Q_2 \cup Q_1) \cup P_2 \cup z_2Xy_2 \cup (z_2Xx_2 \cup x_2x_1)$ is a TK_5 in G' with branch vertices s_1, x_1, y_2, z_1, z_2.

Indeed, $Y_1 = Z_2$. For, if $Y_1 \neq Z_2$ then $Y_2 = Z_1$, $Y_2 - \{s_1, t_1\}$ has a path from y_2 to z_1, and $Y_1 \cup Z_2$ has two independent paths from y_1 to z_2 (since $H - y_2$ is 2-connected). Now these three paths contradict the existence of the cut $\{s_2, t_2\}$ in H.

Then $\{s_2, t_2\} \cap V(Y_1 - \{s_1, t_1\}) \neq \emptyset$. Without loss of generality, we may assume that $t_2 \in V(Y_1) - \{s_1, t_1\}$. Suppose $Y_2 = Z_1$. Then $s_2 \in V(Y_2) - \{s_1, t_1\}$ and we may assume that in H, $\{s_2, t_2\}$ separates $\{s_1, y_1, z_1\}$ from $\{t_1, y_2, z_2\}$. Hence, in Y_1, t_2 separates $\{y_1, s_1\}$ from $\{z_2, t_1\}$, and in Y_2, s_2 separates $\{z_1, s_1\}$ from $\{y_2, t_1\}$. But this contradicts the existence of the paths Y and Z in H. So $Y_2 \neq Z_1$. Since $H - y_2$ is 2-connected and $N_{G'}(y_2) = \{w_1, w_2, w_3, x_1, x_2\}$, we must have $s_2 = w_2 \in \{s_1, t_1\}$. By symmetry, we may assume that $s_2 = w_2 = s_1$.

Let Y_1', Z_2' be the $\{s_2, t_2\}$-bridge of Y_1 containing y_1, z_2, respectively. Then $t_1 \notin V(Z_2')$; for, otherwise, $H - \{s_2, t_2\}$ would contain a path from z_2 to z_1, a contradiction. Therefore, because of the paths Y and Z, $t_1 \in V(Y_1')$ and Y_1' contains disjoint paths R_1, R_2 from $s_2 = s_1, t_1$ to y_1, t_2, respectively. Since $H - y_2$ is 2-connected, Z_1 has independent P_1, Q_1 from z_1 to $s_2 = s_1, t_1$, respectively, and Z_2' has independent paths P_2, Q_2 from z_2 to $s_2 = s_1, t_2$, respectively. Now $z_1Xx_1 \cup z_1Xy_2 \cup y_2x_1 \cup P_1 \cup s_1y_2 \cup (R_1 \cup y_1x_1) \cup P_2 \cup (Q_2 \cup R_2 \cup Q_1) \cup z_2Xy_2 \cup (z_2Xx_2 \cup x_2x_1)$ is a TK_5 in G' with branch vertices s_1, x_1, y_2, z_1, z_2. ∎

Lemma 4.3 *Let $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3, z_1, z_2)$ be an 11-tuple and Y, Z be disjoint paths in $H := G' - V(X - \{y_2, z_1, z_2\} \cup E(X))$ from y_1, z_1 to y_2, z_2, respectively. Then G contains a TK_5 in which y_2 is not a branch vertex or G' contains TK_5, or*

- *(i) there exist $i \in [2]$ and independent paths A, B, C in H, with A and C from z_i to y_1, and B from y_2 to z_{3-i},*
- *(ii) for each $i \in [2]$ satisfying (i), $z_{3-i}x_{3-i} \in E(X)$, and*
- *(iii) H contains two disjoint paths from $V(B - y_2)$ to $V(A \cup C) - \{y_1, z_i\}$ and internally disjoint from $A \cup B \cup C$, with one ending in A and the other ending in C.*

Proof. By Lemma 4.2, we may assume that

- (1) for each $i \in [2]$, H has no path through z_i, z_{3-i}, y_1, y_2 in order (so $y_1z_i \notin E(G)$), and
- (2) there exist $i \in [2]$ and independent paths A, B, C in H, with A and C from z_i to y_1, and B from y_2 to z_{3-i}.

Let $J(A, C)$ denote the $(A \cup C)$-bridge of H containing B, and $L(A, C)$ denote the union of $(A \cup C)$-bridges of H each of which intersects both $A - \{y_1, z_i\}$ and $C - \{y_1, z_i\}$. We choose A, B, C such that the following are satisfied in the order listed:

- (a) A, B, C are induced paths in H,
- (b) whenever possible, $J(A, C) \subseteq L(A, C)$,
- (c) $J(A, C)$ is maximal, and
- (d) $L(A, C)$ is maximal.

We now show that *(ii)* and *(iii)* hold even with the restrictions (a), (b), (c) and (d) above. Let B' denote the union of B and the B-bridges of H not containing $A \cup C$.

- (3) If *(iii)* holds then *(ii)* holds.

Suppose *(iii)* holds. Let $V(P \cap B) = \{p\}$, $V(Q \cap B) = \{q\}$, $V(P \cap C) = \{c\}$ and $V(Q \cap A) = \{a\}$. By the symmetry between A and C, we may assume that y_2, p, q, z_{3-i} occur on B in order. We may further choose P, Q so that pBz_{3-i} is maximal.

To prove *(ii)*, suppose there exists $x \in V(z_{3-i}Xx_{3-i}) - \{x_{3-i}, z_{3-i}\}$. If $N(x) \cap V(H) - \{y_1\} \not\subseteq V(B')$ then G' has a path T from x to $(A - y_1) \cup (C - y_1) \cup (P - p) \cup (Q - a)$ and internally disjoint from $A \cup B' \cup C \cup P \cup Q$; so $A \cup B \cup C \cup P \cup Q \cup T$ contain disjoint paths from y_1, z_i to y_2, x, respectively, contradicting the choice of Y and Z in the 11-tuple (that z_1Xz_2 is maximal). So $N(x) \cap V(H) - \{y_1\} \subseteq V(B')$. Consider $B'' := G[(B' - z_{3-i}) + x]$.

If B'' contains disjoint paths P', Q' from y_2, x to p, q, respectively, then $Q' \cup Q \cup aAz_i$ and $P' \cup P \cup cCy_1$ contradict the choice of Y, Z. If B'' contains disjoint paths P'', Q'' from x, y_2 to p, q, respectively, then $Q'' \cup Q \cup aAy_1$ and $P'' \cup P \cup cCz_i$ contradict the choice of Y, Z.

So we may assume that there is a cut vertex z in B'' separating $\{x, y_2\}$ from $\{p, q\}$. Note that $z \in V(y_2Bp)$.

Since x has at least two neighbors in $B'' - y_2$ (because G is 5-connected and X is induced in $G' - x_1x_2$), the z-bridge of B'' containing $\{x, y_2\}$ has at least three vertices. Therefore, from the maximality of pBz_{3-i} and 2-connectedness of $H - \{y_2, z_1, z_2\}$, there is a path in H from y_1 to $y_2Bz - \{y_2, z\}$ and internally disjoint from $P \cup Q \cup A \cup C \cup B'$. So there is a path Y' in H from y_1 to y_2 and disjoint from $P \cup Q \cup A \cup C \cup pBz_{3-i}$. Now $z_{3-i}Bp \cup P \cup cCz_i \cup A \cup Y'$ is a path in H through z_{3-i}, z_i, y_1, y_2 in order, contradicting (1).

By (2) and (3), it suffices to prove (*iii*). Since $H - \{y_2, z_i\}$ is 2-connected, it contains disjoint paths P, Q from $B - y_2$ to some distinct vertices $s, t \in V(A \cup C) - \{z_i\}$, respectively, and internally disjoint from $A \cup B \cup C$.

(4) We may choose P, Q so that $s \neq y_1$ and $t \neq y_1$.

For, otherwise, $H - \{y_2, z_i\}$ has a separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{v, y_1\}$ for some $v \in V(H)$, $(A \cup C) - z_i \subseteq H_1$ and $B - y_2 \subseteq H_2$. Recall the disjoint paths Y, Z in H from z_1, y_1 to z_2, y_2, respectively. Suppose $v \notin V(Z)$. Then $Z - z_i \subseteq H_2 - \{y_1, v\}$. Hence we may choose Y (by modifying $Y \cap H_1$) so that $V(Y \cap A) = \{y_1\}$ or $V(Y \cap C) = \{y_1\}$. Now $Z \cup A \cup Y$ or $Z \cup C \cup Y$ is a path in H from z_{3-i} to y_2 through z_i, y_1 in order, contradicting (1). So $v \in V(Z)$. Hence $Y \subseteq H_2 - v$, and we may choose Z (by modifying $Z \cap H_1$) so that $V(Z \cap A) = \{z_i\}$ or $V(Z \cap C) = \{z_i\}$. Now $Z \cup A \cup Y$ or $Z \cup C \cup Y$ is a path in H from z_{3-i} to y_2 through z_i, y_1 in order, contradicting (1) and completing the proof of (4).

If $s \in V(A - y_1)$ and $t \in V(C - y_1)$ or $s \in V(C - y_1)$ and $t \in V(A - y_1)$, then P, Q are the desired paths for (*iii*). So we may assume by symmetry that $s, t \in V(C)$. Let $V(P \cap B) = \{p\}$ and $V(Q \cap B) = \{q\}$ such that y_2, p, q, z_{3-i} occur on B in this order. By (1) z_i, s, t, y_1 must occur on C in order. We choose P, Q so that

($*$) sCt is maximal, then pBz_{3-i} is maximal, and then qBz_{3-i} is minimal.

Now consider B', the union of B and the B-bridges of H not containing $A \cup C$. Note that $(P - p) \cup (Q - q)$ is disjoint from B', and every path in H from $A \cup C$ to B' and internally disjoint from $A \cup B' \cup C$ must end in B. For convenience, let $K = P \cup Q \cup A \cup B' \cup C$.

(5) $B' - y_2$ contains independent paths P', Q' from z_{3-i} to p, q, respectively.

Otherwise, $B' - y_2$ has a cut vertex z separating z_{3-i} from $\{p, q\}$. Clearly, $z \in V(qBz_{3-i} - z_{3-i})$, and we choose z so that zBz_{3-i} is minimal.

Let B'' denote the z-bridge of $B' - y_2$ containing z_{3-i}; then $zBz_{3-i} \subseteq B''$. Since $H - \{y_2, z_i\}$ is 2-connected, it contains a path W from some $w' \in V(B'' - z)$ to some $w \in V(P \cup Q \cup A \cup C) - \{z_i\}$ and internally disjoint from K. By the definition of B', $w' \in V(z_iBz_{3-i})$. By (1), $w \notin V(P) \cup V(z_iCt - t)$. By ($*$), $w \notin V(Q) \cup V(tCy_1 - y_1)$.

If $w \in V(A) - \{z_i, y_1\}$ then P, W give the desired paths for (*iii*). So we may assume $w = y_1$ for any choice of W; hence, $z \in V(Z)$ and $Y \cap (B'' \cup (W - y_1)) = \emptyset$. By the minimality of zBz_{3-i}, B'' has independent paths P'', Q'' from z_{3-i} to z, w', respectively. Note that $z_iZz \cap (B'' - z) = \emptyset$. Now $z_iZz \cup P'' \cup Q'' \cup W \cup Y$ is a path in H through z_i, z_{3-i}, y_1, y_2 in order, contradicting (1).

(6) We may assume that $J(A, C) \not\subseteq L(A, C)$.

For, otherwise, there is a path R from B to some $r \in V(A) - \{y_1, z_i\}$ and internally disjoint from $A \cup B' \cup C$. If $R \cap (P \cup Q) \neq \emptyset$, then it is easy to check that $P \cup Q \cup R$ contains the desired paths for (*iii*). So we may assume $R \cap (P \cup Q) = \emptyset$. If $y_2 \notin V(R)$, then P, R are the desired paths for (*iii*). So assume $y_2 \in V(R)$. Recall the paths P', Q' from (5). Then $z_iCs \cup P \cup P' \cup Q' \cup Q \cup tCy_1 \cup y_1Ar \cup R$ is a path in H through z_i, z_{3-i}, y_1, y_2 in order, contradicting (1) and completing the proof of (6).

Let $J = J(A, C) \cup C$. Then by (1), J does not contain disjoint paths from y_2, z_i to y_1, z_{3-i}, respectively. So by Lemma 2.1, there exists a collection $\mathcal{A}$ of subsets of $V(J) - \{y_1, y_2, z_1, z_2\}$ such that $(J, \mathcal{A}, z_i, y_1, z_{3-i}, y_2)$ is 3-planar. We choose $\mathcal{A}$ so that every member of $\mathcal{A}$ is minimal and, subject to this, $|\mathcal{A}|$ is minimum. Then

(7) for any $D \in \mathcal{A}$ and any $v \in V(D)$, $(J[D + N_J(D)], N_J(D) \cup \{v\})$ is not 3-planar.

Suppose for some $D \in \mathcal{A}$ and some $v \in D$, there is a collection of subsets $\mathcal{A}'$ of $D - \{v\}$ such that $(J[D + N_J(D)], \mathcal{A}', N_J(D) \cup \{v\})$ is 3-planar. Then, with $\mathcal{A}'' = (\mathcal{A} - \{D\}) \cup \mathcal{A}'$, $(J, \mathcal{A}'', z_i, y_1, z_{3-i}, y_2)$ is 3-planar. So $\mathcal{A}''$ contradicts the choice of $\mathcal{A}$. Hence, we have (7).

Let $v_1, \ldots, v_k$ be the vertices of $L(A, C) \cap (C - \{y_1, z_i\})$ such that $z_i, v_1, \ldots, v_k, y_1$ occur on C in the order listed. We claim that

(8) $(J, z_i, v_1, \ldots, v_k, y_1, z_{3-i}, y_2)$ is 3-planar.

For, suppose otherwise. Since there is only one C-bridge in J and $(J, \mathcal{A}, z_i, y_1, z_{3-i}, y_2)$ is 3-planar, there exist $j \in [k]$ and $D \in \mathcal{A}$ such that $v_j \in D$. Since H is 2-connected, let $c_1, c_2 \in V(C) \cap N_J(D)$ with c_1Cc_2 maximal.

Suppose $N_J(D) \subseteq V(C)$. Then, since there is only one C-bridge in J and $(J, \mathcal{A}, z_i, y_1, z_{3-i}, y_2)$ is 3-planar, J has a separation (J_1, J_2) such that $V(J_1 \cap J_2) = \{c_1, c_2\}$, $D \cup V(c_1Cc_2) \subseteq V(J_1)$, and $B \subseteq J_2$. Since J has only one C-bridge and C is induced in H, we have $J_1 = c_1Cc_2$. Now let $\mathcal{A}'$ be obtained from $\mathcal{A}$ by removing all members of $\mathcal{A}$ contained in $V(J_1)$. Then $(J, \mathcal{A}', z_i, y_1, z_{3-i}, y_2)$ is 3-planar, contradicting the choice of $\mathcal{A}$.

Thus, let $c \in N_J(D) - V(C)$. So $c \in V(J(A, C))$. Let $D' = J[D + \{c_1, c_2, c\}]$. By (7) and Lemma 2.1, D' contains disjoint paths R from v_j to c and T from c_1 to c_2. We may assume T is induced. Let C' be obtained from C by replacing c_1Cc_2 with T. We now see that A, B, C' satisfy (a), but $J(A, C')$ intersects both $A - \{y_1, z_i\}$ (by definition of v_j and because $c \in V(J(A, C)) - V(C)$) and $C' - \{y_1, z_i\}$ (because of P, Q), contradicting (b) (via (6)) and completing the proof of (8).

(9) There exist disjoint paths R_1, R_2 in $L(A, C)$ from some $r_1, r_2 \in V(C)$ to some $r_1', r_2' \in V(A)$, respectively, and internally disjoint from $A \cup C$, such that z_i, r_1, r_2, y_1 occur on C in this order and z_i, r_2', r_1', y_1 occur on A in this order.

We prove (9) by studying the $(A\cup C)$-bridges of H other than $J(A,C)$. For any $(A\cup C)$-bridge T of H with $T \neq J(A,C)$, if T intersects A let $a_1(T), a_2(T) \in V(T\cap A)$ with $a_1(T)Aa_2(T)$ maximal, and if T intersects C let $c_1(T), c_2(T) \in V(T\cap C)$ with $c_1(T)Cc_2(T)$ maximal. We choose the notation so that $z_i, a_1(T), a_2(T), y_1$ occur on A in order, and $z_i, c_1(T), c_2(T), y_1$ occur on C in order.

If T_1, T_2 are $(A\cup C)$-bridges of H such that $T_2 \subseteq L(A,C)$, $T_1 \neq J(A,C)$, and T_1 intersects C (or A) only, then $c_1(T_1)Cc_2(T_1) - \{c_1(T_1), c_2(T_1)\}$ (or $a_1(T_1)Aa_2(T_1) - \{a_1(T_1), a_2(T_1)\}$) does not intersect T_2. For, otherwise, we may modify C (or A) by replacing $c_1(T_1)Cc_2(T_1)$ (or $a_1(T_1)Aa_2(T_1)$) with an induced path in T_1 from $c_1(T_1)$ to $c_2(T_1)$ (or from $a_1(T_1)$ to $a_2(T_1)$). The new A and C do not affect (a), (b) and (c) but enlarge $L(A,C)$, contradicting (d).

Because of the disjoint paths Y and Z in H, $(H, z_i, y_1, z_{3-i}, y_2)$ is not 3-planar. By (1) $A - \{y_1, z_i\} \neq \emptyset$. Hence, since $H - \{y_2, z_1, z_2\}$ is 2-connected, $L(A,C) \neq \emptyset$. Thus, since $(J, z_i, v_1, \ldots, v_k, y_1, z_{3-i}, y_2)$ is 3-planar (by (8)) and $J(A,C)$ does not intersect $A - \{y_1, z_i\}$ (by (6)), one of the following holds: There exist $(A\cup C)$-bridges T_1, T_2 of H such that $T_1 \cup T_2 \subseteq L(A,C)$, $z_iAa_2(T_1)$ properly contains $z_iAa_1(T_2)$, and $c_1(T_1)Cy_1$ properly contains $c_2(T_2)Cy_1$; or there exists an $(A\cup C)$-bridge T of H such that $T \subseteq L(A,C)$ and $T \cup a_1(T)Aa_2(T) \cup c_1(T)Cc_2(T)$ has disjoint paths from $a_1(T), a_2(T)$ to $c_2(T), c_1(T)$, respectively. In either case, we have (9).

(10) $r_1, r_2 \in V(tCy_1)$ for all choices of R_1, R_2 in (9), or $r_1, r_2 \in V(z_iCs)$ for all choices of R_1, R_2 in (9).

For, suppose there exist R_1, R_2 such that $r_1 \in V(z_iCs)$ and $r_2 \in V(tCy_1)$, or $r_1 \in V(sCt) - \{s,t\}$, or $r_2 \in V(sCt) - \{s,t\}$. Let $A' := z_iAr_2' \cup R_2 \cup r_2Cy_1$ and $C' := z_iCr_1 \cup R_1 \cup r_1'Ay_1$. We may assume A', C' are induced paths in H (by taking induced paths in $H[A']$ and $H[C']$). Note that A', B, C' satisfy (a), and $J(A,C) \subseteq J(A',C')$. However, because of P and Q, $J(A',C')$ intersects both $A' - \{z_i, y_1\}$ and $C' - \{z_i, y_1\}$, contradicting (b) (via (6)) and completing the proof of (10).

If $r_1, r_2 \in V(z_iCs)$ for all choices of R_1, R_2 in (9) then we choose such R_1, R_2 that z_iAr_1' and z_iCr_2 are maximal, and let $z' := r_1'$ and $z'' = r_2$; otherwise, define $z' = z'' = z_i$. Similarly, if $r_1, r_2 \in V(tCy_1)$ for all choices of R_1, R_2 in (9), then we choose such R_1, R_2 that y_1Ar_2' and y_1Cr_1 are maximal, and let $y' := r_2'$ and $y'' = r_1$; otherwise, define $y' = y'' = y_1$. By (10), z_i, z', y', y_1 occur on A in order, and z_i, z'', s, t, y'', y_1 occur on C in order.

Note that H has a path W from some $y \in V(B) \cup V(P-s) \cup V(Q-t)$ to some $w \in V(z_iAz' - \{z', z_i\}) \cup V(z_iCz'' - \{z'', z_i\}) \cup V(y'Ay_1 - \{y', y_1\}) \cup V(y''Cy_1 - \{y'', y_1\})$ such that W is internally disjoint from K. For, otherwise, $(H, z_i, y_1, z_{3-i}, y_2)$ is 3-planar, contradicting the existence of the disjoint paths Y and Z. By (6), $w \notin V(A)$. If $w \in V(z_iAz' - \{z', z_i\}) \cup V(y'Ay_1 - \{y', y_1\})$ then we can find the desired P, Q. So assume $w \in V(z_iCz'' - \{z'', z_i\}) \cup V(y''Cy_1 - \{y'', y_1\})$. By $(*)$ and (1), $y \notin V(B - y_2)$ and $y \notin V(P \cup Q)$. This forces $y = y_2$, which is impossible as $N_H(y_2) = \{w_2\}$. ∎

Remark. Note from the proof of Lemma 4.3 that the conclusions (*ii*) and (*iii*) hold for those paths A, B, C that satisfy (a), (b), (c) and (d).

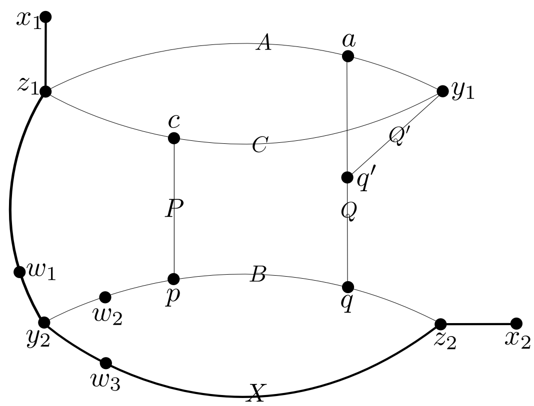

Figure 1: An intermediate structure

5 Finding TK_5

In this section, we prove Theorem 1.1. Let G be a 5-connected nonplanar graph and let $x_1, x_2, y_1, y_2 \in V(G)$ be distinct such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ and $y_1y_2 \notin E(G)$. Let $w_1, w_2, w_3 \in N(y_2) - \{x_1, x_2\}$ be distinct and let $G' := G - \{y_2v : v \notin \{w_1, w_2, w_3, x_1, x_2\}\}$.

We may assume that $G' - x_1x_2$ has an induced path L from x_1 to x_2 such that $y_1, y_2 \notin V(L)$, $(G - y_2) - L$ is 2-connected, and $w_1, w_2, w_3 \in V(L)$; for otherwise, the conclusion of Theorem 1.1 follows from Lemma 3.2. Hence, $G' - x_1x_2$ has an induced path X from x_1 to x_2 such that $y_1 \notin V(X)$, $w_1y_2, w_3y_2 \in E(X)$, and $G' - X = G - X$ is 2-connected. Hence, $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3)$ is a 9-tuple.

We may assume that there exist $z_i \in V(x_iXy_2) - \{x_i, y_2\}$ for $i \in [2]$ such that $H := G' - (X - \{y_2, z_1, z_2\})$ has disjoint paths Y, Z from y_1, z_1 to y_2, z_2, respectively; for, otherwise, the conclusion of Theorem 1.1 follows from Lemma 4.1. We choose such Y, Z so that z_1Xz_2 is maximal. Then $(G, X, x_1, x_2, y_1, y_2, w_1, w_2, w_3, z_1, z_2)$ is an 11-tuple.

By Lemma 4.2 and by symmetry, we may assume that

(1) for $i \in [2]$, H has no path through z_i, z_{3-i}, y_1, y_2 in order (so $y_1z_i \notin E(G)$),

and that there exist independent paths A, B, C in H with A and C from z_1 to y_1, and B from y_2 to z_2. See Figure 1.

Let $J(A, C)$ denote the $(A \cup C)$-bridge of H containing B, and $L(A, C)$ denote the union of $(A \cup C)$-bridges of H intersecting both $A - \{y_1, z_1\}$ and $C - \{y_1, z_1\}$. We may choose A, B, C such that the following are satisfied in the order listed:

(a) A, B, C are induced paths in H,

(b) whenever possible $J(A, C) \subseteq L(A, C)$,

(c) $J(A, C)$ is maximal, and

(d) $L(A, C)$ is maximal.

By Lemma 4.3 and its proof (see the remark at the end of Section 4), we may assume that

$$z_2x_2 \in E(X)$$

and that there exist disjoint paths P, Q in H from $p, q \in V(B - y_2)$ to $c \in V(C) - \{y_1, z_1\}, a \in V(A) - \{y_1, z_1\}$, respectively, and internally disjoint from $A \cup B \cup C$. By symmetry between A and C, we assume that y_2, p, q, z_2 occur on B in order. We further choose A, B, C, P, Q so that

(2) qBz_2 is minimal, then pBz_2 is maximal, and then $aAy_1 \cup cCz_1$ is minimal.

Let B' denote the union of B and the B-bridges of H not containing $A \cup C$. Note that all paths in H from $A \cup C$ to B' and internally disjoint from B' must have an end in B. For convenience, let

$$K := A \cup B' \cup C \cup P \cup Q.$$

Then

(3) H has no path from $aAy_1 - a$ to $z_1Cc - c$ and internally disjoint from K.

For, suppose S is a path in H from some vertex $s \in V(aAy_1 - a)$ to some vertex $s' \in V(z_1Cc - c)$ and internally disjoint from K. Then $z_2Bq \cup Q \cup aAz_1 \cup z_1Cs' \cup S \cup sAy_1 \cup y_1Cc \cup P \cup pBy_2$ is a path in H through z_2, z_1, y_1, y_2 in order, contradicting (1).

We proceed by proving a number of claims from which Theorem 1.1 will follow. Our intermediate goal is to prove (12) that H contains a path from y_1 to $Q - a$ and internally disjoint from K. However, the claims leading to (12) will also be useful when we later consider structure of G near z_1.

(4) $B' - y_2$ has no cut vertex contained in $qBz_2 - z_2$ and, hence, for any $q^* \in V(B') - \{y_2, q\}$, $B' - y_2$ has independent paths P_1, P_2 from z_2 to q, q^*, respectively.

Suppose $B' - y_2$ contains a cut vertex u with $u \in V(qBz_2 - z_2)$. Choose u so that uBz_2 is minimal. Since $H - \{y_2, z_1\}$ is 2-connected, there is a path S in H from some $s' \in V(uBz_2 - u)$ to some $s \in V(A \cup C \cup P \cup Q) - \{p, q\}$ and internally disjoint from K. By the minimality of uBz_2, the u-bridge of $B' - y_2$ containing uBz_2 has independent paths R_1, R_2 from z_2 to s', u, respectively. By the minimality of qBz_2 in (2), S is disjoint from $(P \cup Q \cup A \cup C) - \{z_1, y_1\}$. If $s = z_1$ then $(R_1 \cup S) \cup A \cup (y_1Cc \cup P \cup pBy_2)$ is a path in H through z_2, z_1, y_1, y_2 in order, contradicting (1). So $s = y_1$. Then $(z_1Aa \cup Q \cup qBu \cup R_2) \cup (R_1 \cup S) \cup (y_1Cc \cup P \cup pBy_2)$ is a path in H through z_1, z_2, y_1, y_2 in order, contradicting (1).

Hence, $B' - y_2$ has no cut vertex contained in $qBz_2 - z_2$. Thus, the second half of (4) follows from Menger's theorem.

(5) We may assume that G' has no path from $aAy_1 - a$ to z_1Xz_2 and internally disjoint from $K \cup X$, and no path from $cCy_1 - c$ to $z_1Xz_2 - z_1$ and internally disjoint from $K \cup X$.

For, suppose S is a path in G' from some $s \in V(aAy_1-a) \cup V(cCy_1-c)$ to some $s' \in V(z_1Xz_2)$ and internally disjoint from $K \cup X$, such that $s' \neq z_1$ if $s \in V(cCy_1 - c)$. If $s' = z_1$ then $s \in V(aAy_1-a)$; so $z_2Bq \cup Q \cup aAz_1 \cup S \cup sAy_1 \cup y_1Cc \cup P \cup pBy_2$ is a path in H through z_2, z_1, y_1, y_2 in order, contradicting (1). If $s' = z_2$ then $s = y_1$ by (2); so $(z_1Aa \cup Q \cup qBz_2) \cup S \cup y_1Cc \cup P \cup pBy_2$ is a path in H through z_1, z_2, y_1, y_2 in order, contradicting (1). Hence, $s' \in V(z_1Xz_2) - \{z_1, z_2\}$.

Suppose $s' \in V(z_1Xy_2 - z_1)$. Let P_1, P_2 be the paths in (4) with $q^* = p$. If $s \in V(aAy_1 - a)$ then $z_2x_2 \cup z_2Xy_2 \cup (P_2 \cup P \cup cCy_1) \cup (P_1 \cup Q \cup aAz_1 \cup z_1Xx_1) \cup (y_1As \cup S \cup s'Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. If $s \in V(cAy_1 - c)$ then $z_2x_2 \cup z_2Xy_2 \cup (P_2 \cup P \cup cCz_1 \cup z_1Xx_1) \cup (P_1 \cup Q \cup aAy_1) \cup (y_1Cs \cup S \cup s'Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Now assume $s' \in V(z_2Xy_2 - z_2)$. If $s \in V(aAy_1 - a)$, then $z_1Xx_1 \cup z_1Xy_2 \cup C \cup (z_1Aa \cup Q \cup qBz_2 \cup z_2x_2) \cup (y_1As \cup S \cup s'Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. If $s \in V(cCy_1 - c)$, then $z_1Xx_1 \cup z_1Xy_2 \cup A \cup (z_1Cc \cup P \cup pBz_2 \cup z_2x_2) \cup (y_1Cs \cup S \cup s'Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. This completes the proof of (5).

Denote by $L(A)$ (respectively, $L(C)$) the union of $(A \cup C)$-bridges of H not intersecting C (respectively, A). Let $C' = C \cup L(C)$. The next four claims concern paths from $x_1Xz_1 - z_1$ to other parts of G'. We may assume that

(6) $N(x_1Xz_1 - \{x_1, z_1\}) \subseteq V(C') \cup \{x_1, z_1\}$, and that G' has no disjoint paths from $s_1, s_2 \in V(x_1Xz_1 - z_1)$ to $s_1', s_2' \in V(C)$, respectively, and internally disjoint from $K \cup X$ such that $s_2' \in V(cCy_1 - c)$, x_1, s_1, s_2, z_1 occur on X in order, and z_1, s_1', s_2', y_1 occur on C in order.

First, suppose $N(x_1Xz_1 - \{x_1, z_1\}) \not\subseteq V(C') \cup \{x_1, z_1\}$. Then there exists a path S in G' from some $s \in V(x_1Xz_1) - \{x_1, z_1\}$ to some $s' \in V(A \cup B' \cup P \cup Q) - \{c, y_1, y_2, z_1, z_2\}$ and internally disjoint from $K \cup X$. If $s' \in V(A) - \{z_1, y_1\}$ then $y_1Cc \cup P \cup pBy_2$, $S \cup s'Aa \cup Q \cup qBz_2$ contradict the choice of Y, Z. If $s' \in V(Q - a)$ then $y_1Cc \cup P \cup pBy_2$, $S \cup s'Qq \cup qBz_2$ contradict the choice of Y, Z. If $s' \in V(P - c)$ then let P_1, P_2 be the paths in (4) with $q^* = p$; now $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAy_1) \cup (P_2 \cup pPs' \cup S \cup sXx_1) \cup (C \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. If $s' \in V(B') - \{y_2, p, q\}$ then let P_1, P_2 be the paths in (4) with $q^* = s'$; now $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAy_1) \cup (P_2 \cup S \cup sXx_1) \cup (C \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Now assume G' has disjoint paths S_1, S_2 from $s_1, s_2 \in V(x_1Xz_1 - z_1)$ to $s_1', s_2' \in V(C)$, respectively, and internally disjoint from $K \cup X$ such that $s_2' \in V(cCy_1 - c)$, x_1, s_1, s_2, z_1 occur on X in order, and z_1, s_1', s_2', y_1 occur on C in order. Let P_1, P_2 be the paths in (4) with $q^* = p$. Then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAy_1) \cup (P_2 \cup P \cup cCs_1' \cup S_1 \cup s_1Xx_1) \cup (y_1Cs_2' \cup S_2 \cup s_2Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. This completes the proof of (6).

(7) For any path W in G' from x_1 to some $w \in V(K) - \{y_1, z_1\}$ and internally disjoint from $K \cup X$, we may assume $w \in V(A \cup C) - \{y_1, z_1\}$. (Note that such W exists as G is 5-connected and $G' - X$ is 2-connected.)

For, let W be a path in G' from x_1 to $w \in V(K) - \{y_1, z_1\}$ and internally disjoint from $K \cup X$, such that $w \notin V(A \cup C) - \{z_1, y_1\}$. Then $w \neq y_2$ as $N_{G'}(y_2) = \{w_1, w_2, w_3, x_1, x_2\}$.

Suppose $w \in V(B' - q)$. Let P_1, P_2 be the paths in (4) with $q^* = w$. Then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAy_1) \cup (P_2 \cup W) \cup (C \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

So assume $w \notin V(B' - q)$. Let P_1, P_2 be the paths in (4) with $q^* = p$. If $w \in V(P - c)$ then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAy_1) \cup (P_2 \cup pPw \cup W) \cup (C \cup z_1Xy_2) \cup G[\{x_1, y_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. If $w \in V(Q - a)$ then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup qQw \cup W) \cup (P_2 \cup P \cup cCy_1) \cup (A \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. This completes the proof of (7).

(8) We may assume that G' has no path from $x_1Xz_1 - x_1$ to y_1 and internally disjoint from $K \cup X$.

For, suppose that R is a path in G' from some $x \in V(x_1Xz_1 - x_1)$ to y_1 and internally disjoint from $K \cup X$. Then $x \neq z_1$; as otherwise $z_2Bq \cup Q \cup aAz_1 \cup R \cup y_1Cc \cup P \cup pBy_2$ is a path in H through z_2, z_1, y_1, y_2 in order, contradicting (1). Let P_1, P_2 be the paths in (4) with $q^* = p$. We use W from (7). If $w \in V(A) - \{z_1, y_1\}$ then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAw \cup W) \cup (P_2 \cup P \cup cCy_1) \cup (R \cup xXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. If $w \in V(C) - \{z_1, y_1\}$ then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAy_1) \cup (P_2 \cup P \cup cCw \cup W) \cup (R \cup xXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. This completes the proof of (8).

(9) If G' has a path from $x_1Xz_1 - \{x_1, z_1\}$ to $cCy_1 - c$ and internally disjoint from $K \cup X$, then we may assume that

- $w \in V(C) - \{y_1, z_1\}$ for any choice of W in (7), and
- G' has no path from x_2 to $C - \{y_1, z_1\}$ and internally disjoint from $K \cup X$.

Let S be a path in G' from some $s \in V(x_1Xz_1) - \{x_1, z_1\}$ to $V(cCy_1 - c)$ and internally disjoint from $K \cup X$. Since X is induced in $G' - x_1x_2$, $G'[H - \{y_2, z_1, z_2\} + s]$ is 2-connected. Hence, since $N(x_1Xz_1 - \{x_1, z_1\}) \subseteq V(C') \cup \{x_1, z_1\}$ (by (6)), G' has independent paths S_1, S_2 from s to distinct $s_1, s_2 \in V(C) - \{z_1, y_1\}$ and internally disjoint from $K \cup X$. Because of S, we may assume that z_1, s_1, s_2, y_1 occur on C in this order and $s_2 \in V(cCy_1 - c)$.

Suppose we may choose the W in (7) with $w \in V(A) - \{z_1, y_1\}$. Let P_1, P_2 be the paths in (4) with $q^* = p$. Then $z_2x_2 \cup z_2Xy_2 \cup sXx_1 \cup sXy_2 \cup (P_2 \cup P \cup cCs_1 \cup S_1) \cup (S_2 \cup s_2Cy_1 \cup y_1x_2) \cup (P_1 \cup Q \cup aAw \cup W) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices s, x_1, x_2, y_2, z_2.

Now assume that S' is a path in G' from x_2 to some $s' \in V(C) - \{y_1, z_1\}$ and internally disjoint from $K \cup X$. Then $S_1 \cup S_2 \cup S' \cup (C - z_1)$ contains independent paths S_1', S_2' which are from s to y_1, x_2, respectively (when $s' \in V(z_1Cs_2) - \{s_2, z_1\}$), or from s to c, x_2, respectively (when $s' \in V(s_2Cy_1 - y_1)$). If S_1', S_2' end at y_1, x_2, respectively, then $sXx_1 \cup sXy_2 \cup S_1' \cup S_2' \cup (y_1Aa \cup Q \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices s, x_1, x_2, y_1, y_2. So assume that S_1', S_2' end at c, x_2, respectively. Let P_1, P_2 be the paths in (4) with $q^* = p$. Then $sXx_1 \cup sXy_2 \cup z_2x_2 \cup z_2Xy_2 \cup (S_1' \cup P \cup P_2) \cup S_2' \cup (P_1 \cup Q \cup aAy_1 \cup y_1x_1) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices s, x_1, x_2, y_2, z_2. This completes the proof of (9).

The next two claims deal with $L(A)$ and $L(C)$. First, we may assume that

(10) $L(A) \cap A \subseteq z_1Aa$.

For any $(A \cup C)$-bridge R of H contained in $L(A)$, let $z(R), y(R) \in V(R \cap A)$ such that $z(R)Ay(R)$ is maximal. Suppose for some $(A \cup C)$-bridge R_1 of H contained in $L(A)$, we have $y(R_1)Az(R_1) \not\subseteq z_1Aa$. Let $R_1, \ldots, R_m$ be a maximal sequence of $(A \cup C)$-bridges of H contained in $L(A)$, such that for each $i \in \{2, \ldots, m\}$, R_i contains an internal vertex of $\bigcup_{j=1}^{i-1} z(R_j)Ay(R_j)$ (which is a path). Let $a_1, a_2 \in V(A)$ such that $\bigcup_{j=1}^{m} z(R_j)Ay(R_j) = a_1Aa_2$. By (c), $J(A, C)$ does not intersect $a_1Aa_2 - \{a_1, a_2\}$; so $a_1, a_2 \in V(aAy_1)$. By (d), G' has no path from $a_1Aa_2 - \{a_1, a_2\}$ to C and internally disjoint from $K \cup X$. Hence by (5), $\{a_1, a_2, x_1, x_2, y_2\}$ is a cut in G. Thus, G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a_1, a_2, x_1, x_2, y_2\}$, $P \cup Q \cup B' \cup C \cup X \subseteq G_1$, and $a_1Aa_2 \cup \left(\bigcup_{j=1}^{m} R_j\right) \subseteq G_2$.

Let $z \in V(G_2) - \{a_1, a_2, x_1, x_2, y_2\}$ and assume z_1, a_1, a_2, y_1 occur on A in order. Since G is 5-connected, $G_2 - y_2$ contains four independent paths R_1, R_2, R_3, R_4 from z to x_1, x_2, a_1, a_2, respectively. Now $R_1 \cup R_2 \cup (R_3 \cup a_1Az_1 \cup z_1Xy_2) \cup (R_4 \cup a_2Ay_1) \cup (y_1Cc \cup P \cup pBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z. This completes the proof of (10).

(11) We may assume that if R is an $(A \cup C)$-bridge of H contained in $L(C)$ and $R \cap (cCy_1 - c) \neq \emptyset$ then $|V(R) - V(C)| = 1$ and $N(R - C) = \{c_1, c_2, s_1, s_2, y_2\}$, with $c_1Cc_2 = c_1c_2$ and $s_1s_2 = s_1Xs_2 \subseteq z_1Xx_1$.

For any $(A \cup C)$-bridge R in $L(C)$, let $z(R), y(R) \in V(C \cap R)$ such that $z(R)Cy(R)$ is maximal. Let R_1 be an $(A \cup C)$-bridge of H contained in $L(C)$ such that $R_1 \cap (cCy_1 - c) \neq \emptyset$.

Let $R_1, \ldots, R_m$ be a maximal sequence of $(A \cup C)$-bridges of H contained in $L(C)$, such that for each $i \in \{2, \ldots, m\}$, R_i contains an internal vertex of $\bigcup_{j=1}^{i-1} z(R_j)Cy(R_j)$ (which is a path). Let $c_1, c_2 \in V(C)$ such that $c_1Cc_2 = \bigcup_{j=1}^{m} z(R_j)Cy(R_j)$, with z_1, c_1, c_2, y_1 on C in order. So $c_2 \in V(cCy_1 - y_1)$ and, hence, $c_1 \in V(cCy_1 - y_1)$ by (c) and the existence of P. Let $R' = \bigcup_{j=1}^{m} R_j \cup c_1Cc_2$.

By (c), G' has no path from $c_1Cc_2 - \{c_1, c_2\}$ to $V(B' \cup P \cup Q) \cup \{z_1\}$ and internally disjoint from $K \cup X$. By (d), G' has no path from $c_1Cc_2 - \{c_1, c_2\}$ to $A - \{y_1, z_1\}$ and internally disjoint from $K \cup X$.

If $N(x_2) \cap V(R' - \{c_1, c_2\}) \neq \emptyset$ then by (5) and (9), $N(R' - \{c_1, c_2\}) = \{x_1, x_2, y_2, c_1, c_2\}$. Let $z \in V(R') - \{x_1, x_2, c_1, c_2\}$. Since G is 5-connected, R' has independent paths W_1, W_2, W_3, W_4 from z to x_1, x_2, c_2, c_1, respectively. Now $W_1 \cup W_2 \cup (W_3 \cup c_2Cy_1) \cup (W_4 \cup c_1Cz_1 \cup z_1Xy_2) \cup (y_1Aa \cup Q \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z.

So we may assume $N(x_2) \cap V(R' - \{c_1, c_2\}) = \emptyset$. Since G is 5-connected, it follows from (5) that there exist distinct $s_1, s_2 \in V(x_1Xz_1 - z_1) \cap N(R' - \{c_1, c_2\})$. Choose s_1, s_2 such that s_1Xs_2 is maximal and assume that x_1, s_1, s_2, z_1 occur on X in this order. By (6), $\{c_1, c_2, s_1, s_2, y_2\}$ is a 5-cut in G; so G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{c_1, c_2, s_1, s_2, y_2\}$ and $R' \cup c_1Cc_2 \cup s_1Xs_2 \subseteq G_2$. By (6) again, $(G_2 - y_2, c_1, c_2, s_1, s_2)$ is planar (since G is 5-connected). If $|V(G_2)| \geq 7$ then by Lemma 2.3, (*i*) or (*ii*) or (*iii*) holds. So we may assume that $|V(G_2)| = 6$, and we have the assertion of (11).

We may assume that

(12) H has a path Q' from y_1 to some $q' \in V(Q - a)$ and internally disjoint from K.

First, suppose that $y_1 \in V(J(A, C))$. Then, H has a path Q' from y_1 to some $q' \in V(P - c) \cup V(Q - a) \cup V(B)$ internally disjoint from K. We may assume $q' \in V(P - c) \cup V(B)$;

for otherwise, $q' \in V(Q-a)$ and the claim holds. If $q' \in V(P-c) \cup V(y_2Bq-q)$ then $(P-c)\cup(y_2Bq-q)\cup Q'$ contains a path Q'' from y_1 to y_2; so $z_1Xx_1 \cup z_1Xy_2 \cup C \cup (z_1Aa \cup Q \cup qBz_2 \cup z_2x_2) \cup Q'' \cup G[\{x_1,x_2,y_1,y_2\}]$ is a TK_5 in G' with branch vertices x_1,x_2,y_1,y_2,z_1. Hence, we may assume $q' \in V(qBz_2-q)$. Let P_1,P_2 be the paths in (4) with $q^*=q'$. Then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup Q \cup aAz_1 \cup z_1Xx_1) \cup (P_2 \cup Q') \cup (y_1Cc \cup P \cup pBy_2) \cup G[\{x_1,x_2,y_1,y_2\}]$ is a TK_5 in G' with branch vertices x_1,x_2,y_1,y_2,z_2.

Thus, we may assume that $y_1 \notin V(J(A,C))$. Note that $y_1 \notin V(L(A))$ (by (10)) and $y_1 \notin V(L(C))$ (by (8) and (11)). Hence, since $y_1y_2 \notin E(G)$ and G is 5-connected, y_1 is contained in some $(A\cup C)$-bridge of H, say D_1, with $D_1 \subseteq L(A,C)$ and $D_1 \neq J(A,C)$. Note that $|V(D_1)| \geq 3$ as A and C are induced paths. For any $(A\cup C)$-bridge D of H with that $D \subseteq L(A,C)$ and $D \neq J(A,C)$, let $a(D) \in V(A)\cap V(D)$ and $c(D) \in V(C)\cap V(D)$ such that $z_1Aa(D)$ and $z_1Cc(D)$ are minimal.

Let $D_1,\ldots,D_k$ be a maximal sequence of $(A\cup C)$-bridges of H with $D_i \subseteq L(A,C)$ (so $D_i \neq J(A,C)$) for $i\in[k]$, such that, for each $i \in [k-1]$, $D_{i+1}\cap(A\cup C)$ is not contained in $\bigcup_{j=1}^{i}(c(D_j)Cy_1 \cup a(D_j)Ay_1)$, and $D_{i+1}\cap(A\cup C)$ is not contained in $\bigcap_{j=1}^{i}(z_1Cc(D_j)\cup z_1Aa(D_j))$. Note that for any $i\in[k]$, $\bigcup_{j=1}^{i} a(D_j)Ay_1$ and $\bigcup_{j=1}^{i} c(D_j)Cy_1$ are paths. So let $a_i \in V(A)$ and $c_i \in V(C)$ such that $\bigcup_{j=1}^{i} a(D_j)Ay_1 = a_iAy_1$ and $\bigcup_{j=1}^{i} c(D_j)Cy_1 = c_iCy_1$. Let $S_i = a_iCy_1 \cup c_iCy_1 \cup \left(\bigcup_{j=1}^{i} D_j\right)$.

Next, we claim that for any $l\in[k]$ and for any $r_l \in V(S_l)-\{a_l,c_l\}$ there exist three independent paths A_l,C_l,R_l in S_l from y_1 to a_l,c_l,r_l, respectively. This is clear when $l=1$; note that if $a_l=y_1$, or $c_l=y_1$, or $r_l=y_1$ then A_l, or C_l, or R_l is a trivial path. Now assume that the assertion is true for some $l\in[k-1]$. Let $r_{l+1}\in V(S_{l+1})-\{a_{l+1},c_{l+1}\}$. When $r_{l+1} \in V(S_l)-\{a_l,c_l\}$ let $r_l := r_{l+1}$; otherwise, let $r_l \in V(D_{l+1})$ with $r_l \in V(a_lAy_1-a_l)\cup V(c_lCy_1-c_l)$. By induction hypothesis, there are three independent paths A_l,C_l,R_l in S_l from y_1 to a_l,c_l,r_l, respectively. If $r_{l+1}\in V(S_l)-\{a_l,c_l\}$ then $A_{l+1}:=A_l\cup a_lAa_{l+1}, C_{l+1}:=C_l\cup c_lCc_{l+1}, R_{l+1}:=R_l$ are the desired paths in S_{l+1}. If $r_{l+1}\in V(D_{l+1})-V(A\cup C)$ then let P_{l+1} be a path in D_{l+1} from r_l to r_{l+1} and internally disjoint from $A\cup C$; we see that $A_{l+1}:=A_l\cup a_lAa_{l+1}, C_{l+1}:=C_l\cup c_lCc_{l+1}, R_{l+1}:=R_l\cup P_{l+1}$ are the desired paths in S_{l+1}. So we may assume by symmetry that $r_{l+1}\in V(a_{l+1}Aa_l-a_{l+1})$. Let Q_{l+1} be a path in D_{l+1} from r_l to a_{l+1} and internally disjoint from $A\cup C$. Now $R_{l+1}:=A_l\cup a_lAr_{l+1}, C_{l+1}:=C_l\cup c_lCc_{l+1}, A_{l+1}:=R_l\cup Q_{l+1}$ are the desired paths in S_{l+1}.

We claim that $J(A,C)$ has no vertex in $(a_kAy_1\cup c_kCy_1)-\{a_k,c_k\}$. For, suppose there exists $r\in V(J(A,C))$ such that $r\in V(a_kAy_1-a_k)\cup V(c_kCy_1-c_k)$. Then let A_k,C_k,R_k be independent (induced) paths in S_k from y_1 to a_k,c_k,r, respectively. Let A',C' be obtained from A,C by replacing a_kAy_1, c_kCy_1 with A_k,C_k, respectively. We see that $J(A',C')$ contains $J(A,C)$ and r, contradicting (c).

Therefore, $a\in V(z_1Aa_k)$ and $c\in V(z_1Cc_k)$. Moreover, no $(A\cup C)$-bridge of H in $L(A)$ intersects $a_kAy_1-a_k$ (by (10)). Let S_k' be the union of S_k and all $(A\cup C)$-bridges of H contained in $L(C)$ and intersecting $c_kCy_1-c_k$. Then by (5) and (11), $N(S_k'-\{a_k,c_k\})-\{a_k,c_k,x_2,y_2\}\subseteq V(x_1Xz_1)$. Since G is 5-connected, $N(S_k'-\{a_k,c_k\})-\{a_k,c_k,x_2,y_2\}\neq\emptyset$.

We may assume that $N(S_k'-\{a_k,c_k\})-\{y_2,x_2,a_k,c_k\}\neq\{x_1\}$. For, otherwise, G has a separation (G_1,G_2) such that $V(G_1\cap G_2)=\{a_k,c_k,x_1,x_2,y_2\}$ and $X\cup P\cup Q\subseteq G_1$, and $S_k'\subseteq G_2$. Clearly, $|V(G_1)|\geq 7$. Since G is 5-connected and $y_1y_2\notin E(G)$, $|V(G_2)|\geq 7$. Hence, the assertion follows from Lemma 2.4.

Thus, we may let $z \in N(S'_k - \{a_k, c_k\}) - \{a_k, c_k, x_1, x_2, y_2\}$ such that x_1Xz is maximal. Then $z \neq z_1$. For otherwise, let $r \in V(S'_k) - \{a_k, c_k\}$ such that $rz_1 \in E(G)$. Let $r' = r$ if $r \in V(S_k)$ and, otherwise, let $r' \in V(c_kCy_1 - c_k)$ with $r'r \in E(G)$ (which exists by (11)). Let A_k, C_k, R_k be independent (induced) paths in S_k from y_1 to a_k, c_k, r', respectively. Now $z_2Bq \cup Q \cup aAz_1 \cup (z_1rr' \cup R_k) \cup C_k \cup c_kCc \cup P \cup pBy_2$ is a path in H through z_2, z_1, y_1, y_2 in order, contradicting (1).

Let C^* be the subgraph of G induced by the union of $x_1Xz - x_1$ and the vertices of $L(C) - C$ adjacent to $c_kCy_1 - c_k$ (each of which, by (11), has exactly two neighbors on C and exactly two on x_1Xz_1). Clearly, C^* is connected. Let $G_z = G[x_1Xz \cup S'_k + x_2]$ and let G'_z be the graph obtained from $G_z - \{x_1, x_2\}$ by contracting C^* to a new vertex c^*.

Note that G'_z has no disjoint paths from a_k, c_k to c^*, y_1, respectively; as otherwise, such paths, $c_kCc \cup P \cup pBy_2$, and $a_kAa \cup Q \cup qBz_2$ give two disjoint paths in H which would contradict the choice of Y, Z. Hence, by Lemma 2.1, there exists a collection $\mathcal{A}$ of subsets of $V(G'_z) - \{a_k, c_k, c^*, y_1\}$ such that $(G'_z, \mathcal{A}, a_k, c_k, c^*, y_1)$ is 3-planar. We choose $\mathcal{A}$ so that each member of $\mathcal{A}$ is minimal and, subject to this, $|\mathcal{A}|$ is minimal.

We claim that $\mathcal{A} = \emptyset$. For, let $T \in \mathcal{A}$. By (10), $T \cap V(L(A)) = \emptyset$. Moreover, $T \cap V(L(C)) = \emptyset$; for otherwise, by (11), $c^* \in N(T)$ and $|N(T) \cap V(C)| = 2$; so by (11) again (and since C is induced in H), $(G'_z, \mathcal{A} - \{T\}, a_k, c_k, c^*, y_1)$ is 3-planar, contradicting the choice of $\mathcal{A}$. Thus, $G[T]$ has a component, say T', such that $T' \subseteq L(A, C)$. Hence, for any $t \in V(T')$, $L(A, C)$ has a path from t to $aAy_1 - y_1$ (respectively, $cCy_1 - y_1$) and internally disjoint from $A \cup C$. Since G is 5-connected, $\{x_1, x_2\} \cap N(T') \neq \emptyset$. Therefore, for some $i \in [2]$, G' contains a path from x_i to $aAy_1 - y_1$ as well as a path from x_i to $cCy_1 - y_1$, both internally disjoint from $K \cup X$. However, this contradicts (9).

Hence, $(G'_z, a_k, c_k, c^*, y_1)$ is planar. So by (6) and (11), $(G_z - x_2, a_k, c_k, z, x_1, y_1)$ is planar. By (9) and (10), $N(x_2) \cap V(S_k) \subseteq V(a_kAy_1)$. Therefore, since $(G_z - x_2) - a_kAy_1$ is connected (by (10)), (G_z, a_k, c_k, z, x_2) is planar.

We claim that $\{a_k, c_k, z, x_2, y_2\}$ is a 5-cut in G. For, otherwise, by (7) and (9), G' has a path S_1 from x_1 to $z_1Cc_k - \{z_1, c_k\}$ and internally disjoint from $K \cup X$. However, G' has a path S_2 from z to $c_kXy_1 - c_k$ and internally disjoint from $K \cup X$. Now S_1, S_2 contradict the second part of (6).

Hence, G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a_k, c_k, z, x_2, y_2\}$, $B' \cup P \cup Q \cup X \subseteq G_1$, and $G_z \subseteq G_2$. Clearly, $|V(G_i)| \geq 7$ for $i \in [2]$. So (i) or (ii) or (iii) follows from Lemma 2.3.

Now that we have established (12), the remainder of this proof will make heavy use of Q'. Our next goal is to obtain structure around z_1, which is done using claims (13) – (17). We may assume that

(13) $x_1z_1 \in E(X)$, $w \in V(A) - \{y_1, z_1\}$ for any choice of W in (7), and G' has no path from x_2 to $(A \cup C) - y_1$ and internally disjoint from $K \cup Q' \cup X$.

Let P_1, P_2 be the paths in (4) with $q^* = p$. Suppose $x_1z_1 \notin E(X)$. Let $x_1s \in E(X)$. By (6), G has a path S from s to some $s' \in V(C) - \{y_1, z_1\}$ and internally disjoint from $K \cup Q' \cup X$ (as $Q' \subseteq J(A, C)$). Hence, $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup qQq' \cup Q') \cup (P_2 \cup P \cup cCs' \cup S \cup sx_1) \cup (A \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Now suppose W is a path in (7) ending at $w \in V(C) - \{y_1, z_1\}$. Then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup$

$qQq' \cup Q') \cup (P_2 \cup P \cup cCw \cup W) \cup (A \cup z_1 X y_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Finally, suppose G' has a path S from x_2 to some $s \in V(A \cup C) - \{y_1\}$ and internally disjoint from $K \cup Q' \cup X$. If $s \in V(A - y_1)$ then $z_1x_1 \cup z_1Xy_2 \cup C \cup (z_1As \cup S) \cup (Q' \cup q'Qq \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. If $s \in V(C - y_1)$ then $z_1x_1 \cup z_1Xy_2 \cup A \cup (z_1Cs \cup S) \cup (Q' \cup q'Qq \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

(14) We may assume that G' has no path from y_2Xz_2 to $(A \cup C) - y_1$ and internally disjoint from $K \cup Q' \cup X$, and no path from $y_2Xz_1 - z_1$ to $A - z_1$ and internally disjoint from $K \cup Q' \cup X$.

First, suppose S is a path in G' from some $s \in V(y_2Xz_2)$ to some $s' \in V(A \cup C) - \{y_1\}$ and internally disjoint from $K \cup Q' \cup X$. Then $s \neq y_2$ as $N_{G'}(y_2) = \{w_1, w_2, w_3, x_1, x_2\}$. If $s' \in V(C - y_1)$ then $z_1x_1 \cup z_1Xy_2 \cup A \cup (z_1Cs' \cup S \cup sXx_2) \cup (Q' \cup q'Qq \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. If $s' \in V(A - y_1)$ then $z_1x_1 \cup z_1Xy_2 \cup C \cup (z_1As' \cup S \cup sXx_2) \cup (Q' \cup q'Qq \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

Now suppose S is a path in G' from $s \in V(y_2Xz_1 - z_1)$ to $s' \in V(A - z_1)$ and internally disjoint from $K \cup Q' \cup X$. Let P_1, P_2 be the paths in (4) with $q^* = p$. Then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup qQq' \cup Q') \cup (P_2 \cup P \cup cCz_1 \cup z_1x_1) \cup (y_1As' \cup S \cup sXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

(15) We may assume that

- $J(A, C) \cap (z_1Cc - c) = \emptyset$,
- any path in $J(A, C)$ from $A - \{y_1, z_1\}$ to $(P - c) \cup (Q - a) \cup (Q' - y_1) \cup B$ and internally disjoint from $K \cup Q'$ must end on $(Q \cup Q') - q$, and
- for any $(A \cup C)$-bridge D of H with $D \neq J(A, C)$, if $V(D) \cap V(z_1Cc - c) \neq \emptyset$ and $u \in V(D) \cap V(z_1Ay_1 - z_1)$ then $J(A, C) \cap (z_1Au - \{z_1, u\}) = \emptyset$.

First, suppose there exists $s \in V(J(A, C)) \cap V(z_1Cc - c)$. Then H has a path S from s to some $s' \in V(P - c) \cup V(Q - a) \cup V(Q' - y_1) \cup V(B - y_2)$ and internally disjoint from $K \cup Q'$. If $s' \in V(Q' - y_1) \cup V(Q - a) \cup V(z_2Bp - p)$ then $S \cup (Q' - y_1) \cup (Q - a) \cup (z_2Bp - p)$ contains a path S' from s to z_2; so $S' \cup sCz_1 \cup A \cup y_1Cc \cup P \cup pBy_2$ is a path in H through z_2, z_1, y_1, y_2 in order, contradicting (1). Hence, $s' \in V(P - c) \cup V(y_2Bp - y_2)$ and, by (2), $s = z_1$. Let P_1, P_2 be the paths in (4) with $q^* = p$ (if $s' \in V(P - c)$) or $q^* = s'$ (if $s' \in V(y_2Bp) - \{p, y_2\}$). Then $S \cup (P - c) \cup P_2$ contains a path S' from z_1 to z_2. Let W, w be given as in (7). By (13), $w \in V(A) - \{y_1, z_1\}$. Now $z_2x_2 \cup z_2Xy_2 \cup z_1x_1 \cup z_1Xy_2 \cup S' \cup (P_1 \cup Q \cup aAw \cup W) \cup (C \cup y_1x_2) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_2, z_1, z_2.

Now suppose S is path in $J(A, C)$ from $s \in V(A - \{y_1, z_1\})$ to $s' \in V(P - c) \cup V(B - q)$ and internally disjoint from $K \cup Q'$. Since $N_{G'}(y_2) = \{w_1, w_2, w_3, x_1, x_2\}$, $s' \neq y_2$. Let P_1, P_2 be the paths in (4) with $q^* = p$ (if $s' \in V(P - c)$) or $q^* = s'$ (if $s' \in V(B - q)$). Let S' be a path in $P_2 \cup S \cup (P - c)$ from s to z_2. Let W, w be given as in (7). By (13), $w \in V(A) - \{y_1, z_1\}$. Hence, $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup qQq' \cup Q') \cup (S' \cup sAw \cup W) \cup (C \cup z_1Xy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Finally, suppose D is some $(A\cup C)$-bridge of H with $D \neq J(A,C)$, $v \in V(D)\cap V(z_1Cc-c)$, and $u \in V(D)\cap V(z_1Ay_1 - z_1)$. Then D has a path T from v to u and internally disjoint from $K \cup Q'$. If there exists $s \in V(J(A,C)) \cap V(z_1Au - \{z_1, u\})$ then $J(A,C)$ has a path S from s to some $s' \in V(Q-a)$ and internally disjoint from K. Now $z_2Bq \cup qQs' \cup S \cup sAz_1 \cup z_1Cv \cup T \cup uAy_1 \cup y_1Cc \cup P \cup pBy_2$ is a path in H through z_2, z_1, y_1, y_2 in order, contradicting (1).

(16) We may assume $L(A) = \emptyset$.

Suppose $L(A) \neq \emptyset$. For each $(A\cup C)$-bridge R of H contained in $L(A)$, let $a_1(R), a_2(R) \in V(R\cap A)$ with $a_1(R)Aa_2(R)$ maximal. Let $R_1, \ldots, R_m$ be a maximal sequence of $(A\cup C)$-bridges of H contained in $L(A)$, such that for $i = 2, \ldots, m$, R_i contains an internal vertex of $\bigcup_{j=1}^{i-1}(a_1(R_j)Aa_2(R_j))$ (which is a path). Let $a_1, a_2 \in V(A)$ such that $\bigcup_{j=1}^{m} a_1(R_j)Aa_2(R_j) = a_1Aa_2$. Let $L = \bigcup_{j=1}^{m} R_j$.

By (c), $J(A,C) \cap (a_1Aa_2 - \{a_1,a_2\}) = \emptyset$. By (d), $L(A,C)\cap(a_1Aa_2 - \{a_1,a_2\}) = \emptyset$. By (10), $a_1, a_2 \in V(z_1Aa)$. So $z_1 \notin N(L\cup a_1Aa_2 - \{a_1,a_2\})$. Hence by (14), $V(z_1Xz_2 - y_2) \cap N(L \cup a_1Aa_2 - \{a_1,a_2\}) = \emptyset$. By (13), $x_2 \notin N(L\cup a_1Aa_2 - \{a_1,a_2\})$. Thus, $\{a_1,a_2,x_1,y_2\}$ is a cut in G separating L from X, which is a contradiction (since G is 5-connected).

(17) $z_1c \in E(C)$, $z_1y_2 \in E(G)$, and z_1 has degree 5 in G.

Let C^* be the union of z_1Cc and all $(A\cup C)$-bridges of H intersecting $z_1Cc - c$. By (15), $V(C^*\cap J(A,C)) = \{c\}$.

Suppose (17) fails. If $C^* = z_1Cc$ then, since A, C are induced paths and $L(A) = \emptyset$ (by (16)), $z_1y_2 \in E(G)$ and $z_1Cc \neq z_1c$; so any vertex of $z_1Cc - \{c, z_1\}$ would have degree 2 in G (by (15)), a contradiction. So $C^* - z_1Cc \neq \emptyset$. Since $G' - X$ is 2-connected, $(C^* - z_1Cc)\cap(A - z_1) \neq \emptyset$ by (c) (and since $J(A.C)\cap\cap(zCc-c) = \emptyset$ by (15)). Moreover, if $|V(z_1Cc)| \geq 3$ then there is a path in C^* from $z_1Cc - \{c, z_1\}$ to $A - z_1$ and internally disjoint from $A\cup C$.

Let $a^* \in V(A\cap C^*)$ with a^*Ay_1 minimal, and let $u \in V(z_1Xy_2)$ with uXy_2 minimal such that u is a neighbor of $(C^* - c)\cup(z_1Aa^* - a^*)$.

We may assume that $\{a^*, c, u, x_1, y_2\}$ is a 5-cut in G. First, note, by (15), that $J(A,C)\cap((z_1Aa^* - a^*)\cup(z_1Cc - c)) = \emptyset$ (in particular, $a^* \in V(z_1Aa)$). Hence, if $u = z_1$ then it is clear from (d), (13) and (14) that $\{a^*, c, u, x_1, y_2\}$ is a 5-cut in G. So we may assume $u \neq z_1$. Then G' contains a path T from u to $u' \in V(A - z_1)$ and internally disjoint from $A\cup cCy_1\cup P\cup Q\cup Q'\cup B'$. Suppose $\{a^*, c, u, x_1, y_2\}$ is not a 5-cut in G. Then by (d), (13) and (14), G' has a path R from $r \in V(z_1Xu - u)$ to $r' \in V(P-c)\cup V(Q-a)\cup V(Q'-y_1)\cup V(B')$ and internally disjoint from $K\cup X$. Note that $r' \neq y_2$ as $N_{G'}(y_2) = \{w_1, w_2, w_3, x_1, x_2\}$. If $r' \in V(B' - q)$ then let P_1, P_2 be the paths in (4) with $q^* = r'$; now $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup qQq' \cup Q') \cup (P_2 \cup R \cup rXx_1) \cup (y_1Au' \cup T \cup uXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2. If $r' \in V(P - c)$ then let P_1, P_2 be the paths in (4) with $q^* = p$; now $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup qQq' \cup Q') \cup (P_2 \cup pPr' \cup R \cup rXx_1) \cup (y_1Au' \cup T \cup uXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2. Now assume $r' \in V(Q-a)\cup V(Q'-y_1)$. Then $(Q-a)\cup(Q'-y_1)\cup R$ contains a path R' from r to q. Let P_1, P_2 be the paths in (4) with $q^* = p$; now $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup R' \cup rXx_1) \cup (P_2 \cup P \cup cCy_1) \cup (y_1Au' \cup T \cup uXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_2.

Thus, G has a separation (G_1, G_2) such that $V(G_1\cap G_2) = \{a^*, c, u, x_1, y_2\}$, $uXx_2 \cup P \cup Q \subseteq G_1$, and $C^* \cup z_1Cc \cup z_1Aa^* \subseteq G_2$. Suppose $G_2 - y_2$ contains disjoint paths T_1, T_2 from u, x_1

to a^*, c, respectively. Let P_1, P_2 be the paths in (4) with $q^* = p$. Then $z_2x_2 \cup z_2Xy_2 \cup (P_1 \cup qQq' \cup Q') \cup (P_2 \cup P \cup T_2) \cup (y_1Aa^* \cup T_1 \cup uXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. So we may assume that such T_1, T_2 do not exist. Then by Lemma 2.1, $(G_2 - y_2, u, x_1, a^*, c)$ is planar (as G is 5-connected). If $|V(G_2)| \geq 7$ then, by Lemma 2.3, (i) or (ii) or (iii) holds. Hence, we may assume that $|V(G_2)| = 6$ and, hence, we have (17).

We have now forced a structure around z_1. Next, we study the structure of $G'[B' \cup y_2Xz_2]$ to complete the proof of Theorem 1.1. We may assume that

(18) $(G'[B' \cup y_2Xz_2], p, q, z_2, y_2)$ is 3-planar.

For, otherwise, by Lemma 2.1, $G'[B' \cup y_2Xz_2]$ has disjoint paths R_1, R_2 from q, p to y_2, z_2, respectively. Now $z_1x_1 \cup z_1Xy_2 \cup A \cup (z_1Cc \cup P \cup R_2 \cup z_2x_2) \cup (R_1 \cup qQq' \cup Q') \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. So we may assume (18).

Since G is 5-connected, G is $(5, V(K \cup Q' \cup y_2Xx_2 \cup z_1x_1))$-connected. Recall that $w_1y_2 \in E(x_1Xy_2)$. Then w_1y_2 and w_1Xz_1 are independent paths in G from w_1 to y_2, z_1, respectively. So by Lemma 2.6, G has five independent paths Z_1, Z_2, Z_3, Z_4, Z_5 from w_1 to z_1, y_2, z_3, z_4, z_5, respectively, and internally disjoint from $K \cup Q' \cup y_2Xx_2 \cup z_1x_1$, where $z_3, z_4, z_5 \in V(K \cup Q' \cup y_2Xx_2 \cup z_1x_1)$. Note that we may assume $Z_2 = w_1y_2$. Hence, Z_1, Z_2, Z_3, Z_4, Z_5 are paths in G'. By the fact that X is induced, by (14), and by (5) and (17), $z_3, z_4, z_5 \in V(P) \cup V(Q - a) \cup V(Q') \cup V(B' - y_2)$. Recall that $L(A) = \emptyset$ from (16), and recall W and w from (7) and (13).

(19) We may assume that at least two of Z_3, Z_4, Z_5 end in $B' - y_2$.

First, suppose at least two of Z_3, Z_4, Z_5 end on P. Without loss of generality, let c, z_3, z_4, p occur on P in this order. Let P_1, P_2 be the paths in (4) with $q^* = p$. Then $(Z_1 \cup z_1x_1) \cup Z_2 \cup z_2x_2 \cup z_2Xy_2 \cup (Z_4 \cup z_4Pp \cup P_2) \cup (Z_3 \cup z_3Pc \cup cCy_1 \cup y_1x_2) \cup (P_1 \cup Q \cup aAw \cup W) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_2, z_2.

Now assume at least two of Z_3, Z_4, Z_5 are on $Q \cup Q'$, say Z_3 and Z_4. Then $Z_3 \cup Z_4 \cup Q \cup Q'$ contains two independent paths Z_3', Z_4' from w_1 to z', q, respectively, where $z' \in \{a, y_1\}$. Hence $(Z_1 \cup z_1x_1) \cup Z_2 \cup (Z_3' \cup z'Ay_1) \cup (Z_4' \cup qBz_2 \cup z_2x_2) \cup (y_2Bp \cup P \cup cCy_1) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_1, y_2.

So we may assume that $z_3 \in V(B') - \{p, q\}$, and hence $Z_3 = w_1z_3$. Suppose none of Z_4, Z_5 ends in $B' - y_2$. Then we may assume $z_4 \in V(P - p)$. Let P_1, P_2 be the paths in (4) with $q^* = z_3$. Then $(Z_1 \cup z_1x_1) \cup Z_2 \cup z_2x_2 \cup z_2Xy_2 \cup (Z_3 \cup P_2) \cup (P_1 \cup Q \cup aAw \cup W) \cup (Z_4 \cup z_4Pc \cup cCy_1 \cup y_1x_2) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_2, z_2.

(20) We may assume that

- w_1 has at most one neighbor in B' that is in qBz_2 or separated from y_2Bp in $G'[B' \cup y_2Xz_2]$ by a 2-cut contained in qBz_2, and
- w_1 has at most one neighbor in B' that is in $y_2Bp - y_2$ or separated from qBz_2 in $G'[B' \cup y_2Xz_2]$ by a 2-cut contained in y_2Bp.

Suppose there exist distinct $v_1, v_2 \in N(w_1) \cap V(B')$ such that for $i \in [2]$, $v_i \in V(qBz_2)$ or $G'[B' \cup y_2Xz_2]$ has a 2-cut contained in qBz_2 and separating v_i from y_2Bp. Then, since

$(G'[B' \cup y_2Xz_2], p, q, z_2, y_2)$ is 3-planar (by (18)) and $H - y_2$ is 2-connected, $G'[B' + w_1] - y_2Bp$ contains independent paths S_1, S_2 from w_1 to q, z_2, respectively. Now $w_1Xx_1 \cup w_1y_2 \cup (S_1 \cup qQq' \cup Q') \cup (S_2 \cup z_2x_2) \cup (y_1Cc \cup P \cup pBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_1, y_2.

Now suppose there exist distinct $v_1, v_2 \in N(w_1) \cap V(B')$ such that for $i \in [2]$, $v_i \in V(y_2Bp)$ or $G'[B' \cup y_2Xz_2]$ has a 2-cut contained in y_2Bp and separating v_i from qBz_2. Then, since $(G'[B' \cup y_2Xz_2], p, q, z_2, y_2)$ is 3-planar (by (18)) and $H - y_2$ is 2-connected, $G'[B' + w_1] - (qBz_2 - z_2)$ has independent paths S_1, S_2 from w_1 to p, z_2, respectively. Now $w_1Xx_1 \cup w_1y_2 \cup z_2x_2 \cup z_2Xy_2 \cup S_2 \cup (S_1 \cup P \cup cCy_1 \cup y_1x_2) \cup (z_2Bq \cup Q \cup aAw \cup W) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_2, z_2.

(21) $G'[B' \cup y_2Xz_2]$ has a 2-separation (B_1, B_2) such that $N(w_1) \cap V(B' - y_2) \subseteq V(B_1)$, $pBq \subseteq B_1$, and $y_2Xz_2 \subseteq B_2$.

Let $z \in N(w_1) \cap V(B')$ be arbitrary. If there exists a path S in $B' - (pBy_2 \cup (qBz_2 - z_2))$ from z_2 to z then $z_2x_2 \cup z_2Xy_2 \cup (z_2Bq \cup qQq' \cup Q') \cup (S \cup zw_1 \cup w_1Xx_1) \cup (y_1Cc \cup P \cup pBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. So we may assume that such path S does not exist. Then, since $(G'[B' \cup y_2Xz_2], p, q, z_2, y_2)$ is 3-planar (by (18)) and $G' - X$ is 2-connected, $z \in V(y_2Xp \cup qBz_2)$ (in which case let $B'_z = z$ and $B''_z = G'[B' \cup y_2Xz_2]$), or $G'[B' \cup y_2Xz_2]$ has a 2-separation (B'_z, B''_z) such that $B'_z \cap B''_z \subseteq y_2Bp \cup qBz_2 \cup y_2Xz_2$, $z \in V(B'_z - B''_z)$ and $z_2 \in V(B''_z - B'_z)$.

We claim that we may assume that w_1 has exactly two neighbors in B', say v_1, v_2, such that $v_1 \in V(y_2Bp - y_2)$ or $G'[B' \cup y_2Xz_2]$ has a 2-cut contained in y_2Bp and separating v_1 from qBz_2, and $v_2 \in V(qBz_2 - z_2)$ or $G'[B' \cup y_2Xz_2]$ has a 2-cut contained in qBz_2 and separating v_2 from y_2Bp. This follows from (20) if for every choice of z, $B'_z \cap B''_z \subseteq y_2Bp$ or $B'_z \cap B''_z \subseteq qBz_2$. So we may assume that there exists $v \in N(w_1) \cap V(B')$ such that $pBq \subseteq B'_v$ and we choose v and (B'_v, B''_v) with B'_v maximal. If $pBq \subseteq B'_z$ for all choices of z then, by (18), we have (21). Thus, we may assume that there exists $z \in N(w_1) \cap V(B')$ such that $pBq \not\subseteq B'_z$ for any choice of (B'_z, B''_z). Then $B'_z \cap B''_z \subseteq y_2Bp$ or $B'_z \cap B''_z \subseteq qBz_2$. First, assume $B'_z \cap B''_z \subseteq qBz_2$. Then by the maximality of B'_v, $B' - y_2Bp$ has independent paths T_1, T_2 from z_2 to q, z, respectively. Hence, $z_2x_2 \cup z_2Xy_2 \cup (T_1 \cup qQq' \cup Q') \cup (T_2 \cup zw_1 \cup w_1Xx_1) \cup (y_1Cc \cup P \cup pBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2. Now assume $B'_z \cap B''_z \subseteq y_2Bp$. Then by (20), for any $t \in N(w_1) \cap V(B'_v)$, $t \notin V(y_2Bp - y_2)$ and $G'[B' \cup y_2Xz_2]$ has no 2-cut contained in y_2Bp and separating t from qBz_2. If for every choice of $t \in N(w_1) \cap V(B'_v)$, we have $t \in V(qBz_2 - z_2)$ or $G'[B' \cup y_2Xz_2]$ has a 2-cut contained in qBz_2 and separating t from y_2Bp then the claim follows from (20). Hence, we may assume that t can be chosen so that $t \notin V(qBz_2 - z_2)$ and $G'[B' \cup y_2Xz_2]$ has no 2-cut contained in qBz_2 and separating t from y_2Bp. Then, by (18) and 2-connectedness of $G' - X$, $G[B' + w_1] - (qBz_2 - z_2)$ has independent paths S_1, S_2 from w_1 to p, z_2, respectively. Now $w_1Xx_1 \cup w_1y_2 \cup z_2x_2 \cup z_2Xy_2 \cup S_2 \cup (S_1 \cup P \cup cCy_1 \cup y_1x_2) \cup (z_2Bq \cup Q \cup aAw \cup W) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_2, z_2.

Thus, we may assume that $Z_3 = w_1v_1$, $Z_4 = w_1v_2$, and Z_5 ends at some $v_3 \in V(P \cup Q \cup Q') - \{a, p, q\}$. Suppose $v_3 \in V(P - p)$. Let P_1, P_2 be the paths in (4) with $q^* = v_1$. Then $w_1Xx_1 \cup w_1y_2 \cup z_2x_2 \cup z_2Xy_2 \cup (w_1v_1 \cup P_2) \cup (Z_5 \cup v_3Pc \cup cCy_1 \cup y_1x_2) \cup (P_1 \cup Q \cup aAw \cup W) \cup G[\{x_1, x_2, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_2, z_2.

Now assume $v_3 \in V(Q \cup Q') - \{a, q\}$. Then $(B' - y_2Bp) \cup Z_5 \cup Q \cup Q' \cup (A - z_1) \cup w_1v_2$ has independent paths R_1, R_2 from w_1 to y_1, z_2, respectively. So $w_1Xx_1 \cup w_1y_2 \cup R_1 \cup (R_2 \cup z_2x_2) \cup (y_1Cc \cup P \cup pBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices w_1, x_1, x_2, y_1, y_2. This completes the proof of (21).

By (21), let $V(B_1 \cap B_2) = \{t_1, t_2\}$ with $t_1 \in V(y_2Bp)$ and $t_2 \in V(qBz_2)$. Choose $\{t_1, t_2\}$ so that B_2 is minimal. Then we may assume that $(G'[B_2 + x_2], t_1, t_2, x_2, y_2)$ is 3-planar. For, otherwise, by Lemma 2.1, $G'[B_2 + x_2]$ contains disjoint paths T_1, T_2 from t_1, t_2 to x_2, y_2, respectively. Then $z_1x_1 \cup z_1Xy_2 \cup A \cup (z_1Cc \cup P \cup pBt_1 \cup T_1) \cup (Q' \cup q'Qq \cup qBt_2 \cup T_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1.

Suppose there exists $ss' \in E(G)$ such that $s \in V(z_1Xw_1 - w_1)$ and $s' \in V(B_2) - \{t_1, t_2\}$. Then $s' \notin V(X)$, as X is induced in $G' - x_1x_2$. By (19), (20) and (21), we may assume that $B_1 - qBt_2$ contains a path R from z_3 to p. By the minimality of B_2 and 2-connectedness of $H - y_2$, $(B_2 - t_1) - (y_2Xz_2 - z_2)$ contains independent paths R_1, R_2 from z_2 to s', t_2, respectively. Now $z_2x_2 \cup z_2Xy_2 \cup (R_1 \cup s's \cup sXx_1) \cup (R_2 \cup t_2Bq \cup qQq' \cup Q') \cup (y_1Cc \cup P \cup R \cup z_3w_1y_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_2.

Thus, we may assume that ss' does not exist. Since G is 5-connected, $\{t_1, t_2, y_2, x_2\}$ is not a cut. So H has a path T from some $t \in V(y_2Xx_2) - \{y_2, x_2\}$ to some $t' \in V(P \cup Q \cup Q' \cup A \cup C) - \{p, q\}$ and internally disjoint from $K \cup Q'$. By (14), $t' \notin V(A \cup C) - \{y_1\}$.

If $t' \in V(P - p)$ then $z_1x_1 \cup z_1Xy_2 \cup A \cup (z_1Cc \cup cPt' \cup T \cup tXx_2) \cup (Q' \cup q'Qq \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. So we assume $t' \in V(Q \cup Q') - \{a, q\}$.

If $q \neq q'$ or $t' \in V(Q')$ then $(T \cup Q \cup Q') - q$ has a path Q^* from t to y_1; now $z_1x_1 \cup z_1Xy_2 \cup A \cup (z_1Cc \cup P \cup pBz_2 \cup z_2x_2) \cup (Q^* \cup sXy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. So assume $q = q'$ and $t' \in V(Q) - \{a, q\}$. Then $z_1x_1 \cup z_1Xy_2 \cup C \cup (z_1Aa \cup aQt' \cup T \cup tXx_2) \cup (Q' \cup qBy_2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G' with branch vertices x_1, x_2, y_1, y_2, z_1. ∎

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